Solutions: Homework set #5 (assigned 13 April, due 22 April)

There were a lot of different aspects to this assignment, from deep computations to subtle thinking. I tried to make it somewhat of a synergy of various things we have learned in the course. And I know it was hard, but I hope you learned some cool stuff from it. I encourage you to read this solution set thoroughly, both to see how I did these problems but also to see what kinds of things can be learned by applying geometry and algebra to problems in the Solar System.

The same comments as always apply for this homework: you must show all your work, you must be careful with your units, you must answer all parts of all questions, and so on. Come talk to me if you're not certain what I've done, or why.

Let me also emphasize that I don't expect that any of you turned in homeworks that looked like this – this is way more rigorous and complete than I expected you to do. I have written as complete a solution set as I could to make it as clear as possible what I did. As always, please ask if what I did is not clear. I would be happy to review these solutions and your work with you.

The main problem many people had with their homeworks was unit conversions. You must be very careful with converting centimeters to kilometers and especially m^3 to km^3 (for example). You know that 1000 m = 1 km. Therefore, 1 km^3 must equal $(1000\text{ m})^3$, or 10^9 m. Checking units is an easy thing to do and I encourage you always to do it. Bottom line: remember to be careful and check to make sure your units agree, cancel out, and make sense. I would be more than happy to go over various units issues with you if you like – come see me.

Other common mistakes: Problem 1: many people calculated the time to cross the galaxy in light years or parsecs, both of which are distances. Problem 2: Many people did not put the Earth outside the habitable zone that you calculated.

1) I encourage you to lay out the problem before you really attack it. Write down each question I ask you. Write down what you are trying to find. Write down what you know. Then start from the top and work your way down. You will often find that something you found early in the problem will help you lower down in the problem.

The first thing I want to know is the travel time between a typical very nearby star (assume 20 pc) and the Earth. I know the velocity is 600 km/sec. I also know that 1 parsec is equal to 3×10^{13} km. I want to find the travel time. So I can simply recognize that velocity is distance divided by time, which means that $t = d/v$. You can check that this is right by comparing the units: distance is in kilometers and velocity is in km/sec. Thus, d/v should give us seconds, which is indeed the units for time.

Now I want to put it all together. But first I need to convert distance from parsecs to kilometers. One way to think about this is as follows:

$$20\text{ pc} = 20\text{ pc} \times \left(\frac{3 \times 10^{13}\text{ km}}{1\text{ pc}}\right) = 6 \times 10^{14}\text{ km}.$$

What I have done here is to multiply 20 pc by the quantity in the parentheses, which is equal to 1. Then you can cancel out the units and multiply the numbers out and you find that 20 pc is equal to 6×10^{14} km. Please come see me if this is not clear.

Now we are ready to find our answer:

$$t = \frac{d}{v} = \frac{6 \times 10^{14}\text{ km}}{600\text{ km/sec}} = 10^{12}\text{ sec}.$$

You can calculate how long 10^{12} seconds is using the same kind of units conversion we did above; it turns out to be about 30,000 years or so. This is a surprisingly short time! Surprising to me, anyway. The travel time turns out to be quite small compared to the 100 million years needed for life to originate on Earth. As we will see below, the short travel times just might help you start life elsewhere and transfer it to Earth via panspermia. In this case, we see that if life is arriving from Proxima Centauri, there is plenty of time between the end of Late Heavy Bombardment (LHB) and the first record of life on Earth, 3.8 billion years ago. Even if the stuff arriving from Proxima Centauri is not alive, but just the building blocks of the stuff needed for life, you might argue that you could still have 100 million years for life to originate if the Proxima Centauri impactor arrives just at the end of LHB.

Now we are given a 10 billion year old star. If we assume that life originated on a planet near that star and then traveled to Earth, from where in the galaxy could this life have come? We know that life had to have arrived on the Earth by 3.8 billion years ago, which means that a life-carrying rock had 6.2 billion years to travel from this other star to the Earth. We can rearrange the equation we used above and say $d = v \times t$ (check the units to make sure this makes sense). We can use the travel velocity from above and a time of 6.2 billion years (which is around 2×10^{17} seconds – did you do your units conversion correctly?) to determine that the maximum distance to a 10 billion year old star that could have seeded the Earth with life is around 10^{20} km, or 4 million parsecs. But wait – there is something funny about this number. Recall that the size of the galaxy is around 100 kpc, even if the Earth is all the way at one extreme of the galaxy (which it is not). The answer we get here implies that Earth could been seeded by life from *any* Sun-like star in the galaxy and by stars far outside the galaxy! This answer was a big surprise to me when I first did this problem – when I wrote this problem, I didn't expect that that would be the answer.

Now let us place our ideal 10 billion year old star on the opposite side of the galaxy and let life evolve there. How long does life have to evolve there before it must arrive at Earth? Well, we know life has to be here 3.8 billion years ago, so the amount of time life has to evolve at that other star is equal to 6.2 billion years *minus* the travel time from that star to the Earth. You are all totally confident about calculating travel time now and easily find that the travel time for a distance of 100 kpc and a velocity of 600 km/sec is around 0.16 billion years (160 million years). Thus the amount of time that is allowed for life to evolve near that other star is a whopping 6 billion years! Much, much longer than the 100 million years allowed on Earth. Suddenly this whole panspermia thing is sounding pretty good. But how many stars are there?

I told you that the typical density of Sun-like stars in the very nearby Solar neighborhood might be around 0.01 Sun-like stars per cubic parsec. That means that in 100 cubic parsecs there should be exactly 1 Sun-like star. If my typical nearby volume has a radius of 20 pc then, using the equation I gave you for the volume of sphere, the volume of that spherical bubble in space is around 33,500 pc^3. Thus the number of Sun-like stars in the very nearby Solar neighborhood is this:

$$\text{Number} = \frac{0.01 \text{ Sun like stars}}{\text{pc}^3} \times 33,500 \text{ pc}^3 = 335 \text{ Sun like stars.}$$

Hey, that's quite a few. How about in the entire galaxy? If I assume a sphere with radius of 50 kpc (not actually true, but good enough for this problem), then – using the same technique we just used – we find something like 5×10^{12} Sun-like stars in the galaxy! That's 5 million millions! Quite a lot.

Now we are heading toward Earth, looking down on the Solar System. The total "cross-section" of the Solar System is just the area of a circle whose radius is the 50 AU. The Earth's cross-section is the area of a circle whose radius is the Earth's radius. Recall that the area of a circle is πr^2. From your perspective, the fraction of the Solar System's total cross-section that the Earth makes up is just the ratio of the Earth's cross-sectional area over the Solar System's cross-sectional area:

$$\frac{\pi R_{Earth}^2}{\pi R_{Solar\ System}^2} = \left(\frac{6700 \text{ km}}{50 \text{ AU} \times \left(\frac{1.5\times10^8 \text{ km}}{1 \text{ AU}}\right)} \right)^2 = 8 \times 10^{-13}$$

remembering that 1 AU is 1.5×10^8 km.

By coincidence (perhaps), this turns out to be kind of an interesting number. We guessed the number of eligible stars to be around 5×10^{12}. If you sent 1 life-bearing rock from a planet around each and every of these eligible stars, the number that would hit the Earth would be, given all the naive assumptions we made, around $5 \times 10^{12} \times 8 \times 10^{-13}$, which is around 4! The reason I say this is interesting is the following: if the total number that would hit Earth was much higher, then we might conclude that such panspermia either almost certainly happened (or else that we made some stupid assumptions, which is probably true). If the total number that would hit Earth were much smaller than 1, then we might (naively) conclude that panspermia likely did not happen. But instead we find a number that is pretty close to 1. In other words, the considerations might just work out correctly for you to conclude that panspermia might have happened – once. But, of course, once is all you need.

How likely is all this, anyway? I was looking for you simply to comment on these different ideas and say how likely you think these possibilities are. Any answer that talks about these issues is correct. Don't forget that life still had to originate somewhere, somehow, even if you give it 10 times longer (or more) to do so.

2) This was a complicated problem with a lot of steps. However, for the first part, you knew what the answer should look like, which hopefully helped. As with some other problems, there are several different ways you could do this problem. Here is one, though you may have come up with others that work just fine as well.

The definition of the habitable zone is the location around a star where liquid water is stable. Therefore, the temperature bounds of the habitable zone will be 273 K and 373 K, freezing and boiling. These are the planetary temperatures of interest that we will plug in for T_{pl} in equation 2 from the assignment. Also in equation 2 we will use $A = 0.5$. Lastly, it is going to be too complicated to keep track of $R_\star$ and $T_\star$ separately so let's use equation 3 from the assignment to rewrite equation 2 from the assignment as follows:

$$\frac{R_\star}{R_\odot} = \left(\frac{T_\star}{T_\odot}\right)^{1.3} \tag{1}$$

which leads to

$$R_\star^2 = R_\odot^2 \frac{T_\star^{2.6}}{T_\odot^{2.6}} \tag{2}$$

which is going to be useful. We can plug equation 2 into equation 2 from the assignment to get

$$T_{pl}^4 = \frac{T_\star^{6.6} R_\odot^2 (1 - A)}{4 T_\odot^{2.6} a^2} \tag{3}$$

which can be rearranged to either

$$T_\star^{6.6} = \frac{4a^2 T_{pl}^4}{(1 - A)} \frac{T_\odot^{2.6}}{R_\odot^2} \tag{4}$$

or to

$$a^2 = \frac{T_\star^{6.6} R_\odot^2 (1 - A)}{4 T_\odot^{2.6} T_{pl}^4} \tag{5}$$

depending on what your next step is going to be. This is nothing but algebra, but make sure you understand and can reproduce the steps that I did. I used equation 4 and calculated $T_\star$ for a bunch of different distances. What that means is that I put in different (arbitrary) values for a and used the values I know for T_{pl}, $T_\odot$, A, and $R_\odot$ and solved for $T_\star$. Remember, you also have to use two different values for T_{pl} also. Practically, the way I found the values of $T_\star$ was that I wrote a little computer program. You could have done this problem that way; or using Excel or something like it; or Mathematica; or simply by plugging in values on your calculator. Note: it can be tricky to take the 6.6th root of a number. The way I did this problem was to solve everything on the right side of the equation and end up with $T_\star^{6.6}$ equals some big number. Because taking the 6.6th root of something is the same as raising that big number to the (1/6.6)th power ($(T_\star^{6.6})^{0.15} = T_\star$; you can work it out if you don't believe me), I was then able to find $T_\star$ for all the different values of a that I used.

So now after finding $T_\star$ for a bunch of different a values (and two different T_{pl} values), I can make the habitable zone plot (Figure 1).

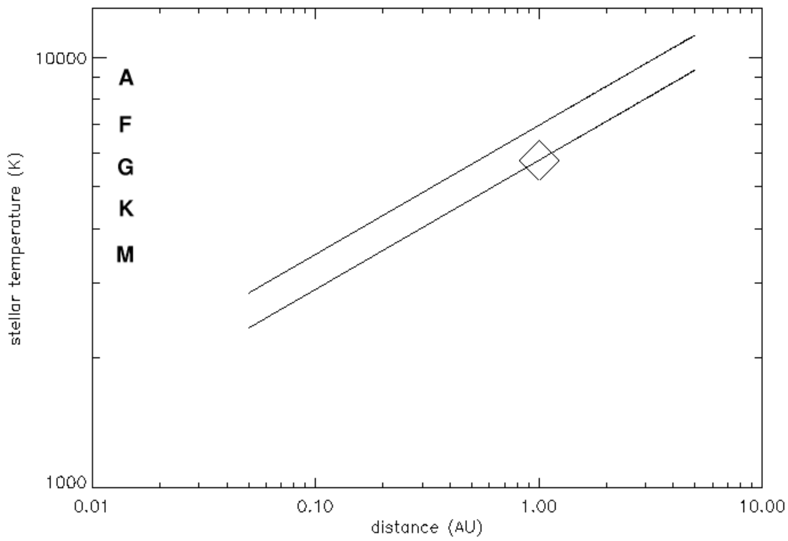

Figure 1: Habitable zone as a function of distance and stellar temperature. The location of the Earth is marked with the diamond. Spectral types of stars are as indicated (from Lecture 18).

The habitable zone plot I made doesn't exactly match the one I showed in lecture because they included more details about stellar properties than we did. Still, it's pretty close. You will also notice that the Earth (1 AU, stellar temperature of 5770 K) does not fall in my habitable zone! This is because we ignored the atmosphere, which adds something like 30 K to the surface temperature due to the greenhouse effect. Also, we used a different albedo than the Earth's actual albedo.

The final piece is to create the continuous habitable zone plot. The luminosity of a star is related to the radius and temperature of the star through equation 5 in the assignment. Thus, I combine equations 4 and 5 from the assignment in the following way

$$L_\star(t) = 4\pi R_\star^2 \sigma T_\star^4 = \frac{L_{\odot,today}}{\left[1 + \frac{2}{5}\left(1 - \frac{t}{t_\odot}\right)\right]} \tag{6}$$

which rearranges and combines with equation 3 from the assignment to give

$$T_\star^{6.6} = \frac{L_{\odot,today} T_\odot^{2.6}}{4\pi\sigma R_\odot^2} \frac{1}{\left[1 + \frac{2}{5}\left(1 - \frac{t}{t_\odot}\right)\right]}. \tag{7}$$

Finally, I insert equation 7 into equation 5 to get

$$a^2 = \frac{R_\odot^2 (1 - A)}{4 T_\odot^{2.6}} \frac{T_\star^{6.6}}{T_{pl}^4} = \frac{R_\odot^2 (1 - A)}{4 T_\odot^{2.6} T_{pl}^4} \frac{L_{\odot,today} T_\odot^{2.6}}{4\pi\sigma R_\odot^2} \frac{1}{\left[1 + \frac{2}{5}\left(1 - \frac{t}{t_\odot}\right)\right]} \tag{8}$$

which simplifies to

$$a^2 = \frac{L_{\odot,today}(1-A)}{16\pi\sigma T_{pl}^4} \frac{1}{\left[1+\frac{2}{5}\left(1-\frac{t}{t_\odot}\right)\right]} \tag{9}$$

as our final equation. So now we have a which is a function of a bunch of constants and of t. Thus, for picking several different values of t, we can produce the appropriate plot (Figure 2). I also made the plot for 10 billion years, for fun (Figure 3).

There is an easier way to do this part. You can rearrange equation 6 to get

$$R_\star^2 T_\star^4 = \frac{L_{\odot,today}}{4\pi\sigma\left[1+\frac{2}{5}\left(1-\frac{t}{t_\odot}\right)\right]} \tag{10}$$

which should look familiar: equation 2 from the assignment also has $R_\star^2 T_\star^4$ in it. I can plug the right side of equation 10 into equation 2 from the assignment and, after rearranging, you get the same thing as equation 9 without having to deal with equation 3 from the assignment.

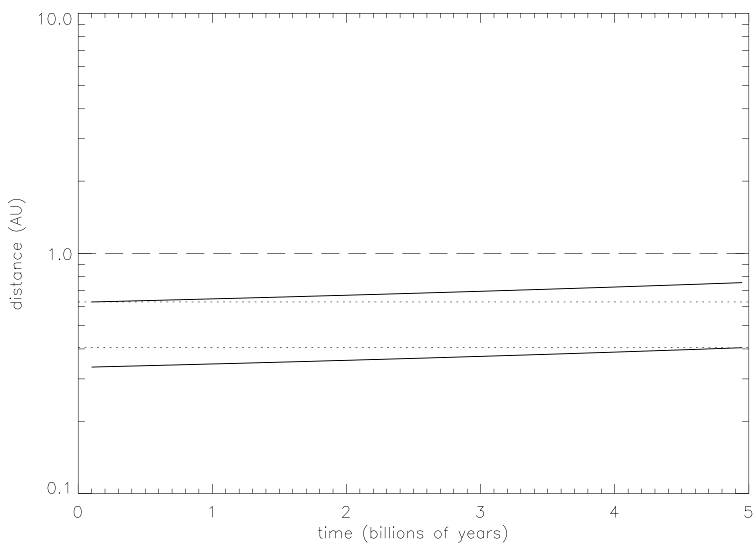

Figure 2: Location of the habitable zone with time. The solid lines indicate the inner and outer edges of the habitable zone. Within the dotted lines is the continuous habitable zone, which here is 0.23 AU wide and centered on 0.51 AU. The dashed line indicates the location of the Earth.

The point of all this was to convince you that you actually know enough – perhaps barring one or two arithmatic accidents – to derive the habitable zone and its location as a function of distance, stellar type and temperature, and time. This is a fundamental piece of astrobiology! As I mentioned above, adding an atmosphere back in to the problem makes things a lot more complicated; the greenhouse effect makes the planet hotter than it otherwise would be, but there are also some feedback effects which are quite complicated. In any case, by including the Earth's atmosphere, we would find that the Earth is indeed presently in the habitable zone – thank goodness! – but that it does not reside in the Solar System's continuous habitable zone. Thus, Jim Kasting and others invoked a thick, early CO_2 atmosphere that caused a larger greenhouse effect and allowed liquid water to exist on the surface of the Earth.

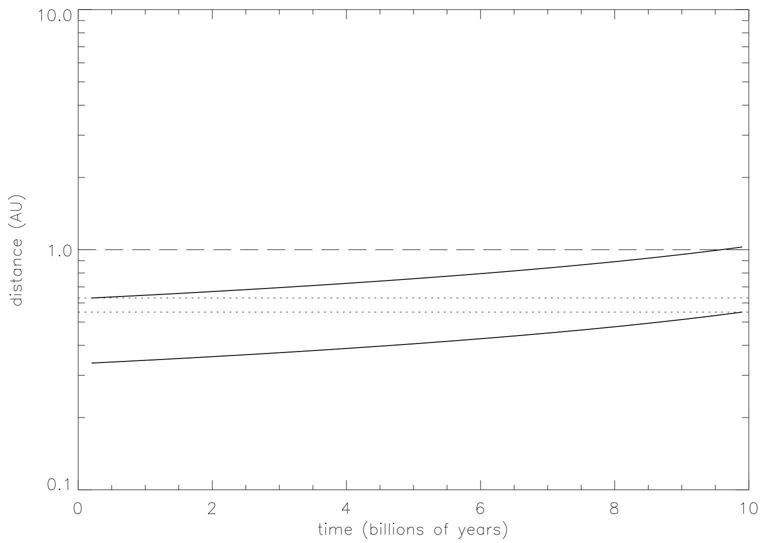

Figure 3: Location of the habitable zone with time. The solid lines indicated the inner and outer edges of the habitable zone. Within the dotted lines is the continuous habitable zone, which here is 0.08 AU wide and centered on 0.59 AU. The dashed line indicates the location of the Earth.

3) Obviously, there is no “right” answer for this question — any creative effort got you credit. Notable answers given on the web page.