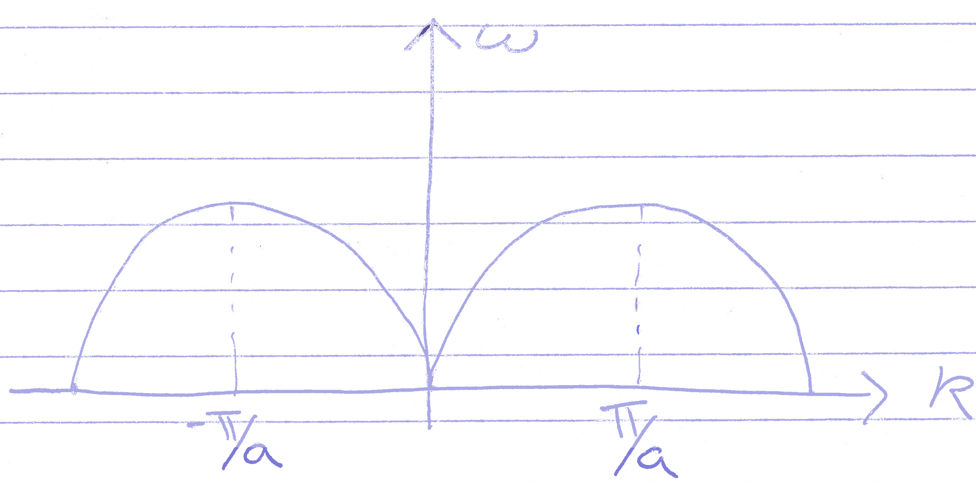
Wed. Oct. 03

- Last time we discovered that simply considering the discrete nature of solids (consisting of atoms), the dispersion becomes non-linear

$$\omega = 2\sqrt{\frac{\alpha}{M}}\left|\sin\left(\frac{ka}{2}\right)\right|$$

where $\alpha \equiv$ spring constant; $M \equiv$ mass of atom

- For a "normal mode" all atoms vibrate at the same frequency, ω
- plot of this dispersion relation:



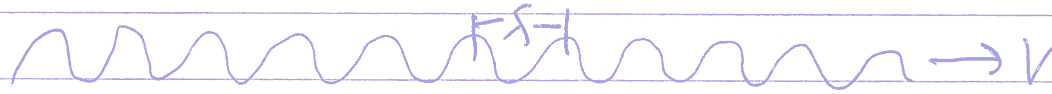
- let's spend some time discussing this relation

(1) only frequencies $0 \le \omega \le \omega_m$ are allowed or possible, where ω_m is the highest frequency at $k = \pm \pi/a$

- at the other extreme, $k \to 0 \Rightarrow \omega \to 0$ or $\lambda \to \infty$, and we recover the result of a continuous medium (approx. linear)

- for small k, $\sin(ka/2) \approx ka/2$ ∴

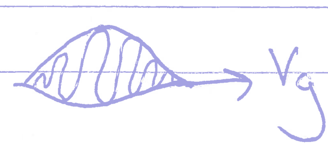
(2) a single wave given by k (wave #) extends to $\pm\infty$ + travels with sound speed V



$k = 2\pi/\lambda$

For a single wave, we only need to know the phase velocity (speed)

- a sound wave will almost always be better represented by a pulse



- moves with group velocity V_g

- from Fourier analysis, a single pulse cannot be described by a single wave number or wavelength

- We require a range of wavelengths with more needed with decreasing pulse width

- the wave packet moves at speed V_g
- different modes move @ different speeds → interference

(3) the group velocity → 0 as the wavelength → 2a !, where 'a' is the periodicity of the lattice

- for $k = \pm \pi/a \Rightarrow \lambda = 2a$

$$\frac{\partial \omega}{\partial k} = 0 \text{ @ } k = \pm \pi/a$$

- we find a standing wave @ $k = \pm \pi/a$

- When $\lambda = 2a$, this is known as the Bragg condition → wave interacts with lattice when condition is met

(4) A system in real space with periodicity 'a' is periodic in reciprocal space with periodicity $2\pi/a$

- in other words, the system is symmetric in both spaces
- real lattice : $x \to x+a$
- reciprocal lattice : $k \to k + 2\pi/a$
- importantly, we only need the region

$$-\pi/a \leq k \leq \pi/a$$

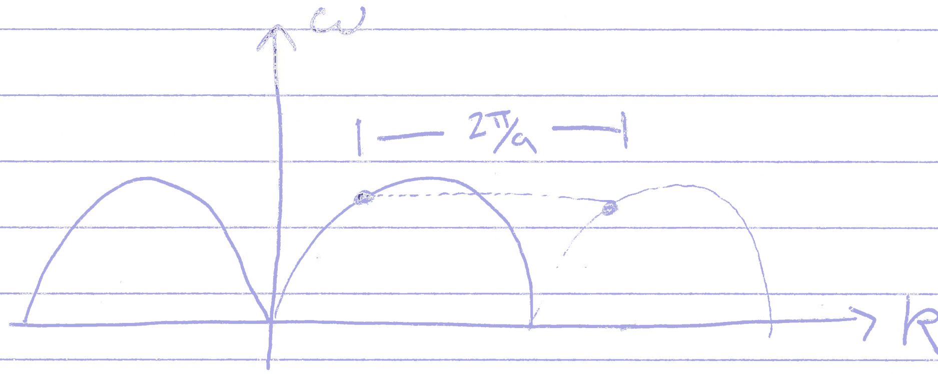
to describe all possible modes (for the present case)

- this is called the "first Brillouin zone"
- English speakes often say

"Brew-yawn"

- visually, what happens when we translate

$$k \to k + 2\pi/a$$



- we see that $\omega(k) = \omega(k + 2\pi/a)$ for all k's
- We already discovered our eq. of motion

$$U_n = Ae^{i(kna - \omega t)}$$

let $k \to k + 2\pi/a$

$$U_n = Ae^{i([k+2\pi/a]na - \omega t)}$$

$$= Ae^{i(kna + 2\pi n - \omega t)}$$

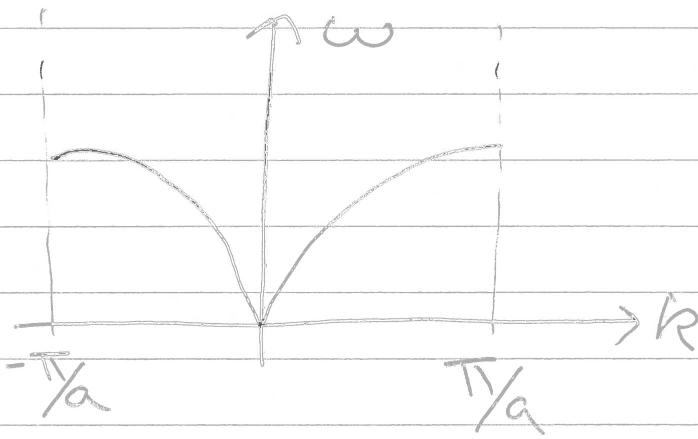
$$= Ae^{i(kna - \omega t)} e^{i2\pi n} = Ae^{i(kna - \omega t)} = U_n$$

$$e^{i2\pi n} = \cos(2\pi n) + i\sin(2\pi n) = 1 \text{ for all } n \in \mathbb{Z}$$

- The take-home message is the following:

  modes with wavelengths shorter than 2a have no physical meaning

- Therefore, all modes are contained in the first Brillouin zone



- It is important to remember that the above was a classical treatment, but the energy values for vibrating atoms is quantized

$$E_n = \hbar\omega(1/2 + n)$$

for the quantum harmonic oscillator