Ch. 5 Magnetostatics

- phenomenon which arrises from currents + mag. mom.
- mag. dipole moment → 1^{st} term in multipole expansion for a general B-field
- mag. quantum numbers + spins determine how atoms respond to fields + are responsible for such phenomena as ferromagnets or perm. mag.
- this is classical E + M, we look at charges in motion to describe magnetism.

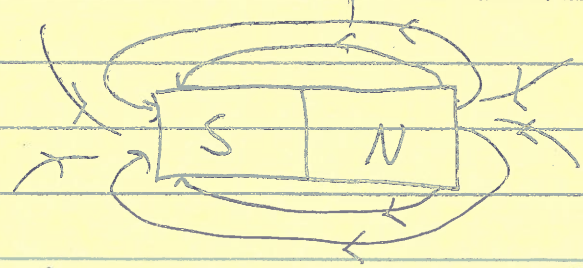
- Magnetic field - also vector field

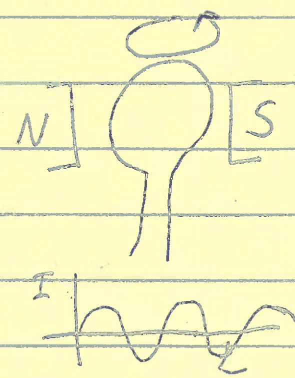
$$\vec{B} + \vec{H}, \text{ Tesla + Amp/meter, resp.}$$

- in relativity E + B are a single entity

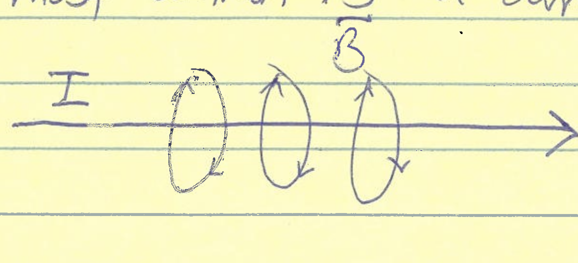
technology!
- electric motors + generators use perm. mag.
- hard drives

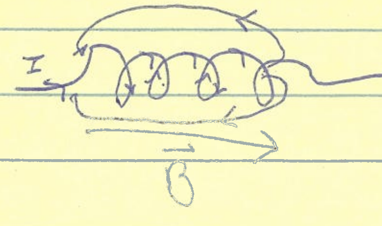
- most common is a permanent magnet





- next most common is a current-carrying wire

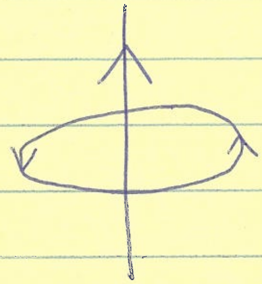




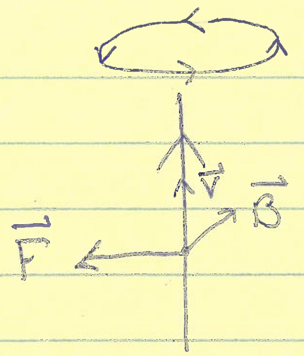
- single point charge

- Beg. Sept. 8th

III



wire 1



wire 2

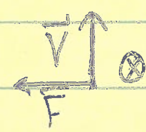
Current $= I = Q/t$

~ drift speed of $1e^-$ at 1 amp ~ 1 meter/hr.

I. *Auxiliary Fields:

$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$; $\vec{P} = \epsilon_0 \chi_e \vec{E}$, $\vec{D} = \epsilon \vec{E}$

$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$; $\vec{M} = \chi_m \vec{H}$, $\vec{H} = \frac{1}{\mu}\vec{B}$



II - magnetic forces:

$\vec{F}_{elec} = Q\vec{E}$

$\vec{F}_{mag} = Q(\vec{v} \times \vec{B})$: "Lorentz force"

*$T = \frac{N \cdot s}{C \cdot m}$

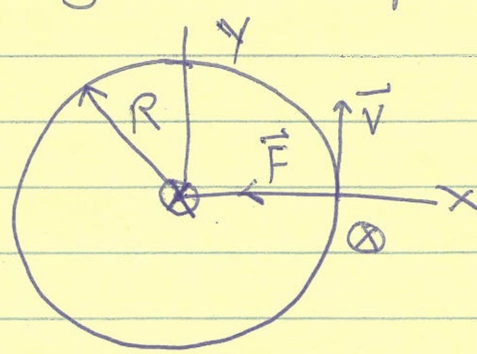
- with $\vec{E}$ + $\vec{B}$, $\vec{F} = Q(\vec{E} + (\vec{v} \times \vec{B}))$

- fundamental axiom → laboratory

4. Ex: 5.1 Cyclotron motion

motion of

- charged particle in a B-field is circular
- B-field gives centripetal acceleration
-



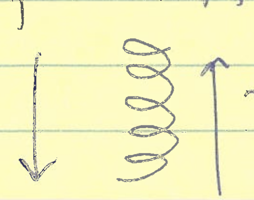
$F = QvB$

$QvB = \frac{mv^2}{R}$

$p = QBR$

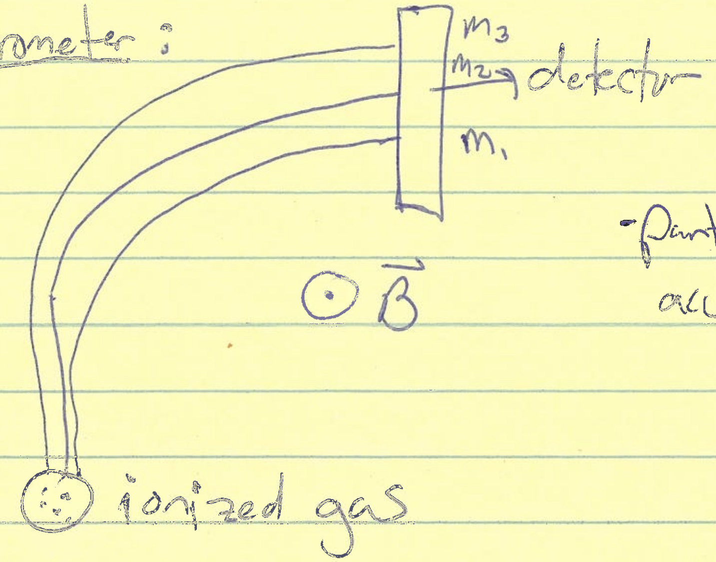
"cyclotron formula"

- 1st modern particle accelerator



$v_{\parallel} \to B$

5 - mass spectrometer :



- particle accelerator

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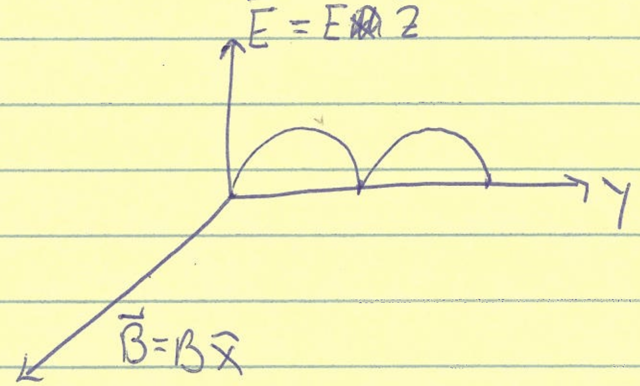
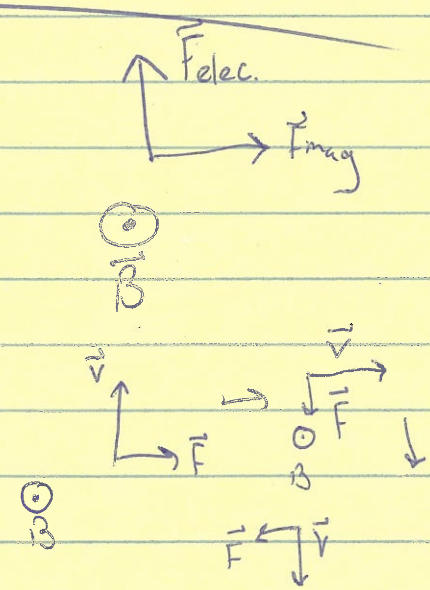




- no force in x direction

$\vec{v} = (0, \dot{y}, \dot{z})$

$$\vec{v} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \dot{y} & \dot{z} \\ B & 0 & 0 \end{vmatrix} = B\dot{z}\hat{y} - B\dot{y}\hat{z}$$

$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B}) = Q(E\hat{z} + B\dot{z}\hat{y} - B\dot{y}\hat{z}) =$$

$$= m\vec{a} = m(\ddot{y}\hat{y} + \ddot{z}\hat{z})$$

$\omega \to \frac{CT}{m}$

$\to \frac{C\,kg\,m\,s}{s^2\,C\,m}\frac{1}{kg} \to \frac{1}{s}$

- Seperate $\hat{y}$ + $\hat{z}$ components

$QB\dot{z} = m\ddot{y}$, $QE - QB\dot{y} = m\ddot{z}$

end mond.

- let $\omega \equiv \frac{QB}{m}$ $\to$ ("cyclotron frequency") (in absense of E-field)

1

- $\ddot{y} = \omega\dot{z}$, $\ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right)$, coupled diff. eq.
  - diff. first, eliminate $\ddot{z}$

$\Rightarrow y(t) = C_1 \cos\omega t + C_2 \sin\omega t + (E/B)t + C_3$

$z(t) = C_2 \cos\omega t - C_1 \sin\omega t + C_4$

- at rest initially $\dot{y}(0) = \dot{z}(0) = 0$; $y(0) = z(0) = 0$

$\Rightarrow$ $y(t) = \frac{E}{\omega B}(\omega t - \sin\omega t)$; $z(t) = \frac{E}{\omega B}(1 - \cos\omega t)$

$R \equiv \frac{E}{\omega B}$; $\sin^2\omega t + \cos^2\omega t = 1$

$\Rightarrow (y - R\omega t)^2 + (z - R)^2 = R^2$

- circle of radius R @ center $(0, R\omega t, R)$

at speed $u = \omega R = \frac{E}{B}$ $\to$ in y direction

- same formula as spot on the rim of a wheel.

4

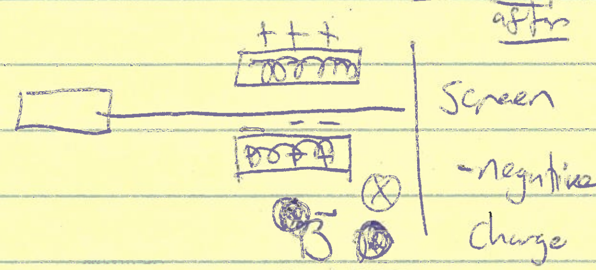
- Magnetic forces do no work
- Q moves $d\vec{l} = \vec{v}\,dt$
- Work $dW_{mag} = \vec{F}_{mag}\cdot d\vec{l} = Q(\vec{v}\times\vec{B})\cdot\vec{v}\,dt = 0$
- $\vec{v}\times\vec{B}$ $\perp$ to $\vec{v}$, so $(\vec{v}\times\vec{B})\cdot\vec{v} = 0$
- B-fields can alter direction, but cannot change its momentum

→ do example after

7

6

Problem 5.3 : pg. 216



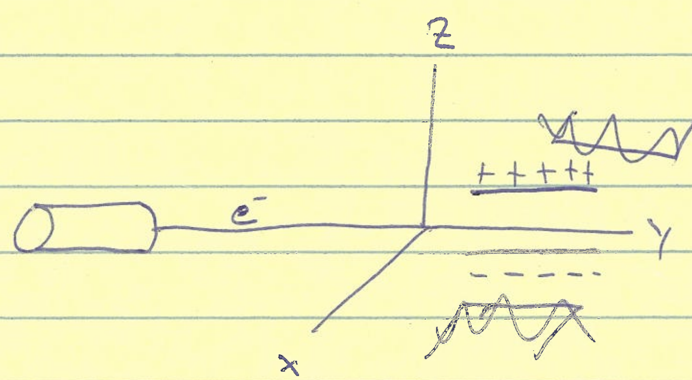
- ask them to do prob.

(a) $\vec{F} = q\left[\vec{E} + (\vec{v}\times\vec{B})\right] = 0$

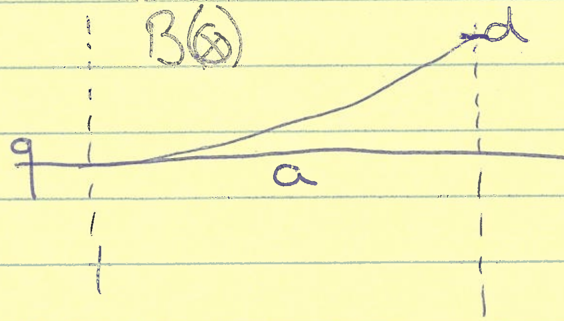
$\Rightarrow E - vB = 0 \Rightarrow \boxed{v = \frac{E}{B}}$

(b) $mv = qBR \Rightarrow \frac{q}{m} = \frac{v}{BR} = \frac{E}{B^2R}$

* $QvB = \frac{mv^2}{R}$



(6) **5.1:**

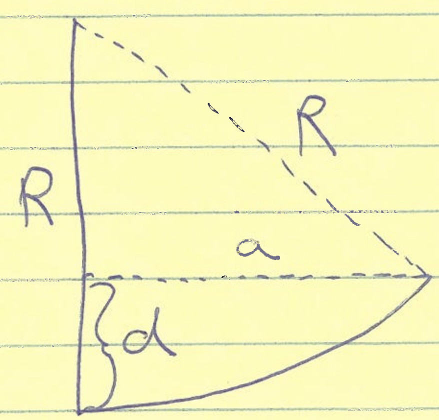


$\longrightarrow \vec{v}$

$\vec{B}\otimes$

is the charge + or − ? (+)

- find the momentum: $p = qBR$, in terms of a, d, B, q



- need to find $\underline{R}$
- know q, B

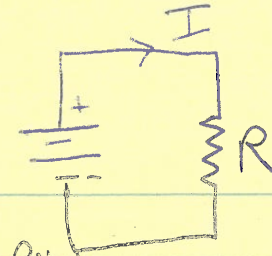
$$(R-d)^2 + a^2 = R^2$$

$$\Rightarrow R^2 - 2Rd + d^2 + a^2 = R^2 \Rightarrow R = \frac{a^2+d^2}{2d}$$

$$p = qBR = \boxed{qB\left(\frac{a^2+d^2}{2d}\right) = p}$$

Q: - Is the charge + or - ?

- Sing. elect. moves @ 10^6 m/s
- drift velocity ~1 mm/s

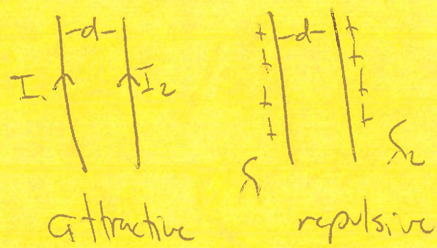


3

- Current : charge per unit time, passing a point
  - wi...
- ele... ...ing, but opposite to ...
- do... ...q in +v direction is ... direction (depends upon prod...)

$F = -\frac{\mu_0 I_1 I_2}{2\pi d}$ → same direction attractive

$F = \frac{\lambda_1 \lambda_2}{2\pi \epsilon_0 d}$



- Charges are scalar, currents vectors

- 1 ... speed , $\lambda = \frac{q}{l}$

- $l = v\Delta t$, carrys charge; $q = \underbrace{\lambda v \Delta t}_{l}$, passes p... $\Rightarrow \boxed{\frac{q}{\Delta t} = \lambda v = I}$

- print HW's ✓
- Quiz announcement #2 - take home → Wed. Sept. 17th due Friday Sept. 19 - beginning of class
- old quizes
- HW handback
- attendance

- ... confined to the wire, ... nitude, 3 referenced

- ... current

$\vec{F}_{mag} = q(\vec{v} \times \vec{B})$

$q = \lambda l$

$... q = \int (\vec{v} \times \vec{B}) \lambda dl = \int (\lambda \vec{v} \times \vec{B}) dl$

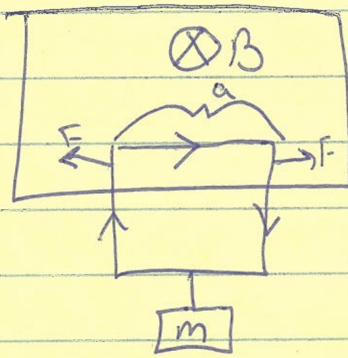
$= \int (\vec{I} \times \vec{B}) dl \Rightarrow \int I (d\vec{l} \times \vec{B})$

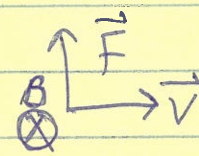
because I + $d\vec{l}$ in same direction

$\Rightarrow \vec{F}_{mag} = I \int (d\vec{l} \times \vec{B})$, since I (magnitude) is constant along wire.

- go to mag. forces do no work (2 pages back)

- Ex: 5.3





$\vec{F}_{horiz} = 0$

$\downarrow mg = \vec{F}_{grav}$

$\vec{F}_{mag} = I \int d\vec{l} \times \vec{B}$

$(\vec{I} \times \vec{B})$ points upward,

$F_{mag} = I B a$, $\vec{F}_{mag} = I \int (d\vec{l} \times \vec{B}) = I \int_0^a dl B$

$\sum F_{vert} = 0$

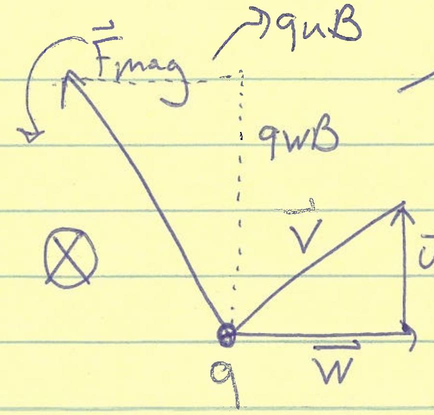
$mg = IBa \Rightarrow \boxed{I = \frac{mg}{Ba}}$

- increase current, rises by h $\rightarrow \sum F_{vert} \neq 0$

- $W_{mag} \neq F_{mag} \cdot h \neq IBah$? <u>no</u>

- Charges have a vertical component:

→ Lorentz



→ amount of force in vert. direction is the same

- horiz. pushes back, but does not change $\vec{w}$, instead lifts mass up.

$w \to$ horizontal component, $u \to$ vertical

$$F_{\text{vert, mag}} = \underbrace{IBa}_{=\lambda awB} \quad I = \lambda w$$

, just as before (no work vs. gravity)

$$\to \text{now } F_{\text{horiz}} = \underbrace{\lambda a}_{\substack{\lambda w \\ q}} uB \qquad \text{in } dt = w\,dt$$

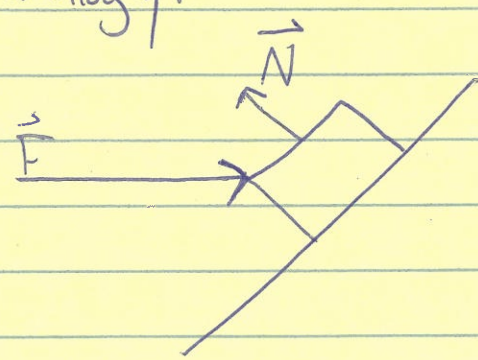
$$W_{\text{battery}} = \lambda a B \int uw\,dt = IBah$$

$$w\lambda = I, \quad \int u\,dt = h$$

$$W = \int F \cdot dl \quad , \quad dl = w\,dt$$

$$\int u\,dt = h, \quad \lambda a B w = IaB$$

- mechanical analogy:



- $\vec{N}$ - does no work, translates $\vec{F}$ into a vertical component

~~- Prob. 5.2 : ⑧~~

~~- Magnetic field of a star~~

Prob. 5.2

2 (a) $\dot{y} = (E/B)\,\hat{y}$

- in $\hat{z}$ direction $\vec{F}_{elec.} = q\vec{E} \rightarrow +z$

$\vec{F}_{mag} = qvB = q\left(\frac{E}{B}\right)B = q\vec{E} \rightarrow -z$

$\left.\right\}$ $z(t) = 0$, $y(t) = \left(\frac{E}{B}\right)t$

- Units of B + E, $E \rightarrow \frac{N}{C}$, $T \rightarrow \frac{N}{C} \cdot \frac{s}{m}$ $\left.\right\}$ $v \rightarrow \frac{m}{s} = \frac{E}{B}$

Tesla

- a particle of 1 C, moving through a B-field at 1 m/s perp. to field experiences 1 N

B-field star: magnetic field, 100 million to 100 billion T (Magnetar) - type of neutron star, Earth ~30 micro T.

II - even at 1000 km, B-field would be lethal

- a hydrogen atom becomes narrow by 200 times

- proton superconductor phase

I · magnetohydrodynamic process (not well understood)

- ~~detected by gamma ray bursts~~

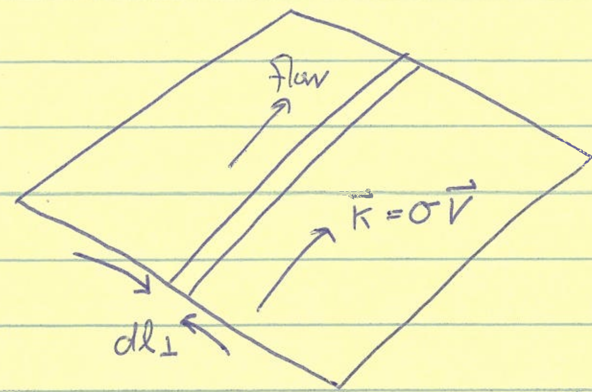
- stars consists of plasmas (charges) in motion

- solar flare → release of mag. energy.

- charge flows over a surface

- Surface current density: $\vec{K} = \sigma \vec{v}$
  - ribbon

$$\int K \, dl_\perp = I$$



$$\vec{K} = \frac{dI}{dl_\perp}$$

$dl_\perp \rightarrow$ infinitesimal width, $\vec{K}$ varies over surface

$$\vec{F}_{mag} = \int_S (\vec{v} \times \vec{B}) \sigma \, da = \int_S (\vec{K} \times \vec{B}) \, da$$

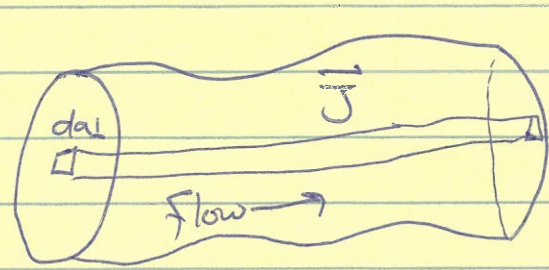
$$\rightarrow \int_S \sigma \, da = q$$

end Fri. Sept 12

- for 3-D case, we have a volume current density, $\vec{J}$.

  - tube of infinitesimal cross-section $da_\perp$

$$\vec{J} = \frac{dI}{da_\perp}$$



$$\vec{J} = \rho \vec{v}$$

$$\vec{F}_{mag} = \int_{vol} (\vec{v} \times \vec{B}) \rho \, d\tau = \int_{vol} (\vec{J} \times \vec{B}) \, d\tau$$

- Ex 5.4

- Current I is uniformly distributed over a wire of cross-section with radius a

- area perp. to flow $= \pi a^2$

$I = J \cdot A$; $J = \dfrac{I}{\pi a^2}$

- suppose $J = ks$, find the total current

$$\vec{J} = \frac{d\vec{I}}{da_\perp} \Rightarrow \vec{I} = \int J\,(s\,ds\,d\phi)$$

$$= \int_0^a 2\pi k s^2\,ds = \frac{2\pi k a^3}{3}$$

- Total current crossing surface S is

$$I = \int_S J\,da_\perp = \int_S \vec{J} \cdot d\vec{a}$$

$$\oint_S \vec{J} \cdot d\vec{a} = \int_{vol} (\nabla \cdot \vec{J})\,d\tau$$ → charge per unit time leaving volume

what if surface is closed?

- Conservation of charge

-what flows out through the surface comes at the expense of charge inside of volume

$-I = \frac{dq}{dt}$

$q = \int \rho \, d\tau$

$$\int_{vol} (\nabla \cdot \vec{J}) \, d\tau = -\frac{d}{dt} \int_{vol} \rho \, d\tau = -\int_{vol} \left(\frac{\partial \rho}{\partial t} \right) d\tau$$

out flow decreases charge in V

Leibniz integral rule

$$\Longrightarrow \nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}, \text{ at some point}$$

"continuity equation"

Notes:

- divergence is the magnitude of a vector field's source or sink
- divergence theorem → sum of sources - sum all sinks → gives the flow out of a volume

$$\oint_S \vec{J} \cdot d\vec{a} = \int_{vol} (\nabla \cdot \vec{J}) \, d\tau$$

sum of sources + sinks

flow out of volume

pg. 222

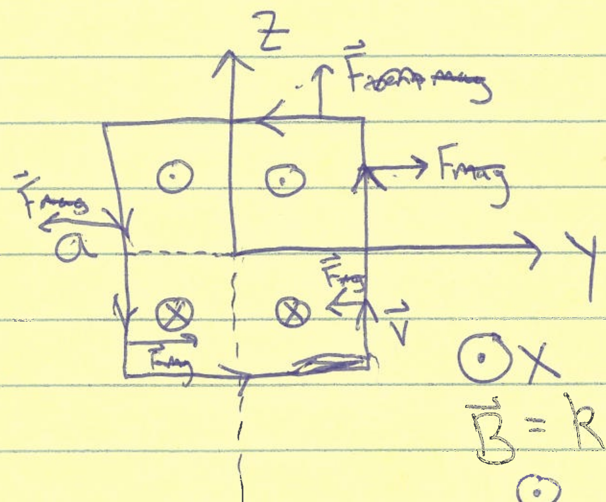
- have them do.

<u>5.4</u>: Suppose B-field has the form

$$\vec{B} = kz\,\hat{x} \quad , \quad \vec{F}_{mag} = \int (\vec{I}\times\vec{B})\,dl = I\int (d\vec{l}\times\vec{B})$$

- find the force on a square loop (side a) in y-z plane, carries current I, counter clockwise

$\vec{I} = \lambda \vec{v}$



Fmag,y = 0

⊙ X

$\vec{B} = kz\hat{x}$

⊙

{kz}

$$F_{z,top} = k\cdot\left(\frac{a}{2}\right)\cdot Ia = \frac{ka^2 I}{2}\,\hat{z}$$

$$F_{z,bottom} = +k\left(\frac{-a}{2}\right)Ia = -\frac{ka^2 I}{2}\,(-\hat{z})$$

$$\Rightarrow \vec{F} = F_z\,\hat{z} = (ka^2 I)\,\hat{z}$$

problem 5.6 :

(a) phonograph record carries surface density σ (uniform)

- rotates at angular velocity ω
- calculate the surface current density, K

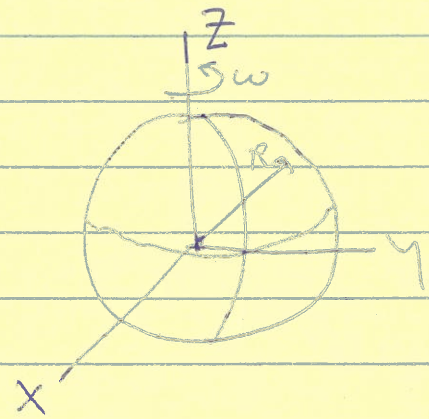
* Current w/ out conductor!

$V = \omega r$,

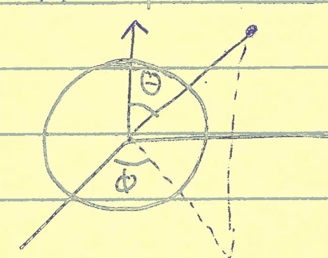
$K = v\sigma$

$\Rightarrow K = \sigma \omega r \rightarrow$ function of r

(b) Solid sphere, radius R, total charge Q



$\vec{V} = \omega r \sin\theta \hat{\phi}$



- find the current density $\vec{J} = J(r, \theta, \phi)$

$$\Rightarrow \vec{J} = \rho \omega r \sin\theta \hat{\phi}$$

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3}$$

$$\nabla \cdot \vec{J} = \frac{1}{\sin\theta} \frac{\partial (J_\phi)}{\partial \phi} = 0$$

$\Rightarrow \partial\rho/\partial t = 0$.

end. Mond. Sept. 15th

The Biot-Savart Law:

- steady currents: give constant B-fields

electro/magnetostatics $\frac{\partial \rho}{\partial t} = 0, \frac{\partial \vec{J}}{\partial t} = 0$

- in wire, magnitude $|\vec{I}|$ must be constant

current density

- also, by the continuity equation $\nabla \cdot \vec{J} = 0$

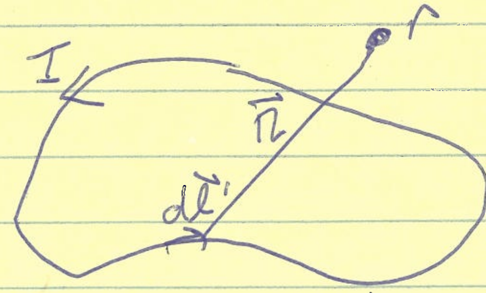
- B-field of a steady current (line) → simplest case is line

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{\imath}}{\imath^2} \, dl' = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}' \times \hat{\imath}}{\imath^2}$$

- along current path along with flow (integral)

$\mu_0 = 4\pi \times 10^{-7} \, N/A^2$; $1T = \frac{N}{A \cdot m}$

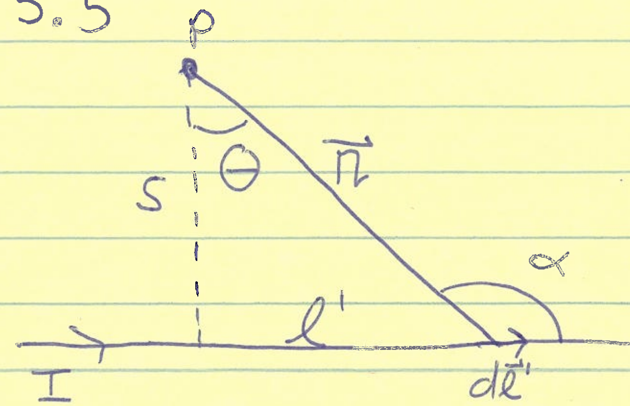
vacuum permeability



- notice the $\frac{1}{r^2}$ proportionality, inverse square dependance

$\theta + \phi + 90° = 180°$
$\alpha + \phi = 180°$
$\Rightarrow \alpha + (90° - \theta) = 180°$
$\therefore \alpha = \theta + 90°$
$\sin\alpha = \sin(\theta + 90°)$
$= \cos\theta$

Ex: 5.5



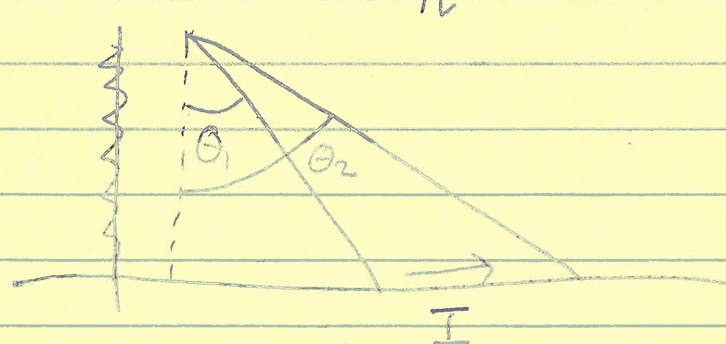
$(d\vec{l}' \times \hat{\imath})$ points out of the page

$|d\vec{l}' \times \vec{\imath}| = dl' \sin\alpha = dl' \cos\theta \quad \Rightarrow \sin\alpha = \cos\theta$

$l' = s\tan\theta \;,\; \dfrac{dl'}{d\theta} = \dfrac{s}{\cos^2\theta}\, d\theta \Big\} \; dl'\cos\theta = \left(\dfrac{s}{\cos^2\theta}\right)\cos\theta\, d\theta$

$\cos\theta = \dfrac{s}{\imath} \Rightarrow \dfrac{1}{\imath^2} = \dfrac{\cos^2\theta}{s^2}$

$$\Rightarrow B = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2} \underbrace{\left(\frac{\cos^2\theta}{s^2}\right)}_{\frac{1}{\imath^2}} \underbrace{\left(\frac{s}{\cos^2\theta}\right)\cos\theta\, d\theta}_{dl'\cos\theta}$$



$$B = \frac{\mu_0 I}{4\pi s}\int_{\theta_1}^{\theta_2} \cos\theta \, d\theta = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)$$

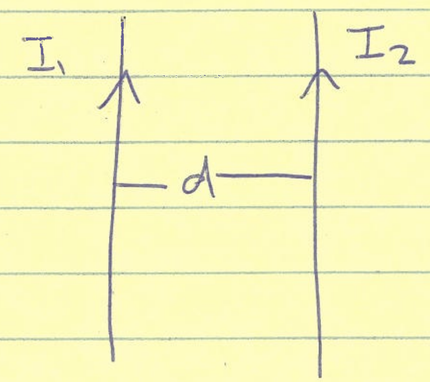
- field of line segment carrying I

- infinite wire : $\theta_1 = -\pi/2$, $\theta_2 = +\pi/2$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi s}$$

- circles around

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$



- field at 2, due to 1

$$B = \frac{\mu_0 I_1}{2\pi d} = B(s=d)$$

$$\vec{F}_{mag} = I\int(d\vec{l} \times \vec{B})$$

- force on wire 2 by wire 1

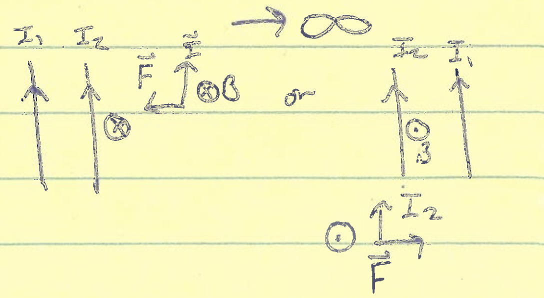
$$|\vec{F}| = I_2 \overbrace{\left(\frac{\mu_0 I_1}{2\pi d}\right)}^{B} \int dl$$

$$f = \frac{|F|}{l} = \frac{\mu_0 I_1 I_2}{2\pi d}$$

- direction

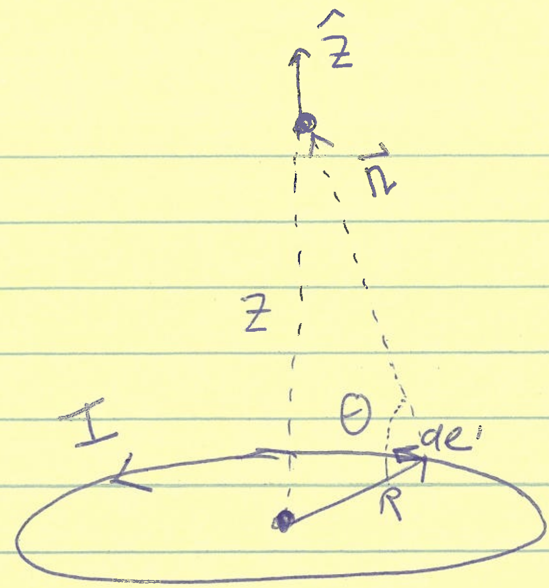
- $\underline{I_1}$ causes $\underline{\text{field}}$

$\to \infty$

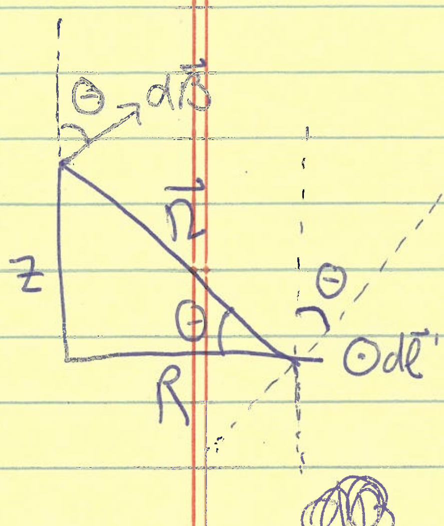


- attractive

Ex: 5.6



$d\vec{B}\cdot\hat{z} = |d\vec{B}|\cos\theta$



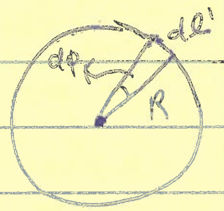
$\vec{B} = \int \frac{\mu_0 I}{4\pi} \frac{d\vec{l}' \times \hat{n}}{n^2}$

- horizontal components cancel

$B(z) = \frac{\mu_0 I}{4\pi} \int \frac{dl'}{n^2} \underbrace{\cos\theta}_{d\vec{B}\cdot\hat{z}}$; $d\vec{l}' \times \hat{n} = dl'$ ↘ perp.

- $\cos\theta$ projects out vertical component

$\Rightarrow B(z) = \frac{\mu_0 I}{4\pi}\left(\frac{\cos\theta}{n^2}\right)\int_0^{2\pi} R\,d\phi'$ - know from symmetry



$\sin\phi \approx \phi$

$|\vec{n}|^2 = R^2 + z^2$, $\cos\theta = \frac{R}{n}$ → both constant

$\Rightarrow B(z) = \frac{\mu_0 I}{4\pi} \cdot \left(\frac{R}{n}\right)\left(\quad\right) 2\pi R \frac{1}{n^2}$

$= \frac{\mu_0 I}{2} \frac{R^2}{(R^2+z^2)^{3/2}}$

- Biot-Savart Law becomes

- surfaces

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, da'$$

$$\vec{K} \equiv \frac{d\vec{I}}{dl_\perp},\ \sigma\vec{v}$$

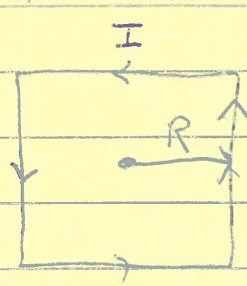
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

$$\vec{J} \equiv \frac{d\vec{I}}{da_\perp},\ \rho\vec{v}$$

- leave $\vec{K}$ + $\vec{J}$ in integral, not constant in general → in line current I is constant

<u>Problem 5.8</u>: pg. 228

(a) B-field at center of square loop

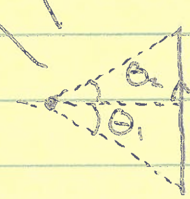


$$B = \frac{\mu_0 I}{4\pi s}\left[\sin\theta_2 - \sin\theta_1\right]$$

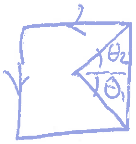
one side: $B = \frac{\mu_0 I}{4\pi s}\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right]$

→ $\theta_2 = \pi/4$, $\theta_1 = -\pi/4$, $4\left(\frac{\sqrt{2}}{2} - \left(\frac{-\sqrt{2}}{2}\right)\right)\frac{\mu_0 I}{4\pi R} = \boxed{\frac{\sqrt{2}\,\mu_0 I}{\pi R}}$

- out



end Wed. Sept. 17th



$\frac{\mu_0 I}{4\pi s}[\sin\theta_2 - \sin\theta_1]$

$\uparrow \pi/4 \qquad \uparrow -\pi/4$

$\rightarrow \frac{\sqrt{2}\,\mu_0 I}{\pi R}$

Prob. 5.8 (b)

- n-sided polygon, carrying steady I, I single side $\frac{\mu_0 I}{4\pi R}[\sin\theta_2 - \sin\theta_1]$

$\$ \quad z = R \quad \theta_2 = \pi/n, \quad \theta_1 = -\pi/n$

- n sides : $B = \frac{n\mu_0 I}{2\pi R}\sin(\pi/n)$, $n=4$: $\frac{2\mu_0 I}{\pi R}\cdot\frac{\sqrt{2}}{2} = \frac{\mu_0 I\sqrt{2}}{\pi R}$

- $\sin(x)$ is odd $\rightarrow \sin(-x) = -\sin(x)$

(c) formula reduces to field at loop center (circle)

$B(z) = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}$, $B(z=0) = \frac{\mu_0 I}{2}\cdot\frac{R^2}{R^3} = \frac{\mu_0 I}{2R} \rightarrow (z=0)$

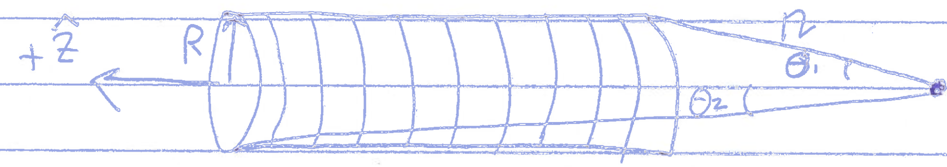
from ex. 5.6

$\lim_{n\to\infty} B = \lim_{n\to 0}\frac{n\mu_0 I}{2\pi R}\sin\left(\frac{\pi}{n}\right)$, $\theta \rightarrow 0$, $\sin\theta \approx \theta$ } 1st order approx

$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

$\rightarrow \frac{n\mu_0 I}{2\pi R}\cdot\left(\frac{\pi}{n}\right) = \frac{\mu_0 I}{2R}$ ✓

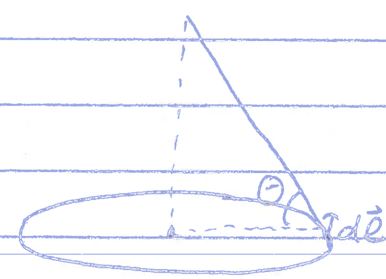
Problem 5.11, page 229



$n \equiv \frac{\text{turns}}{\text{length}}$

- assume turns are circular

- from ex. 5.6



$$\Rightarrow B(z) = \frac{\mu_0 I}{2} \frac{R^2}{(R^2+z^2)^{3/2}}$$

- assume each loop is of width dz

- current in 1 loop = $nI\,dz$ (of dz)

$$B = \frac{\mu_0 n I}{2} \int \frac{R^2}{(R^2+z^2)^{3/2}} dz$$

- polar coordinates:

$$z = R\cot\theta \, , \quad dz = \frac{-R}{\sin^2\theta} d\theta$$

$$\frac{1}{(R^2+z^2)^{3/2}} = \frac{\sin^3\theta}{R^3} \, , \quad \sin\theta = \frac{R}{r} \, , \quad r = \sqrt{R^2+z^2}$$

$$\sin^3\theta = \frac{R^3}{r^3} = \frac{R^3}{(R^2+z^2)^{3/2}}$$

$$\Rightarrow \frac{\sin^3\theta}{R^3} = \frac{1}{(R^2+z^2)^{3/2}}$$

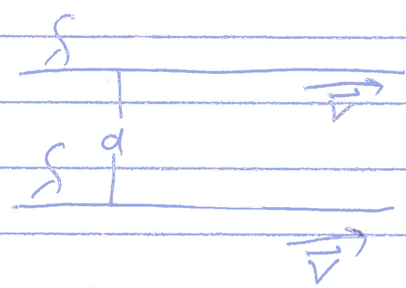
$$B = \frac{\mu_0 n I}{2} \int \frac{R^2 \sin^3\theta}{R^3 \sin^2\theta} (-R\,d\theta)$$

$$= -\frac{\mu_0 n I}{2} \int_{\theta_1}^{\theta_2} \sin\theta \, d\theta = \frac{\mu_0 n I}{2} \cos\theta \Big|_{\theta_1}^{\theta_2}$$

- infinite solenoid: $\theta_2 = 0$, $\theta_1 = \pi$ $= \underbrace{\cos(\theta_2) - \cos(\theta_1)}_{1-(-1)} = 2$

$$\Rightarrow \boxed{B = \mu_0 n I \hat{z}} \checkmark$$

Problem: 5.13, pg. 229



- how great would v have to be for F_{elect} & F_B to balance?

$$E = \frac{\lambda}{2\pi\epsilon_0 s} \rightarrow \text{repulsive} \qquad F = \frac{\lambda}{2\pi\epsilon_0 s} \cdot (\underbrace{\lambda l}_{q})$$

$$\frac{F}{l} = f = \frac{\lambda^2}{2\pi\epsilon_0} \frac{1}{d}$$

- Tr

$$f_m = \frac{\mu_0}{2\pi}\frac{\lambda^2 v^2}{d}, \quad I = \lambda v$$

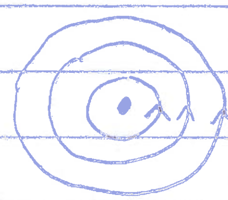
$$f_e = f_m \Rightarrow \frac{1}{2\pi\epsilon_0}\frac{\lambda^2}{d} = \frac{\mu_0}{2\pi}\frac{\lambda^2 v^2}{d}$$

$$\Rightarrow \mu_0 v^2 = \frac{1}{\epsilon_0} \Rightarrow v = \frac{1}{\sqrt{\epsilon_0\mu_0}} = 3\times10^8\, m/s$$

End. Friday Sept. 19th

- Field of an infinite wire

$$\vec{B} = \mu_0 I / 2\pi s \; \hat{\phi}$$



- divergence + curl necessary to determine field
- $\vec{\nabla}\times\vec{E} = 0$, $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$
- $\vec{\nabla}\times\vec{B} = ?$, $\vec{\nabla}\cdot\vec{B} = 0$

- clearly has a non-zero curl, $\vec{\nabla}\times\vec{B} \neq 0$

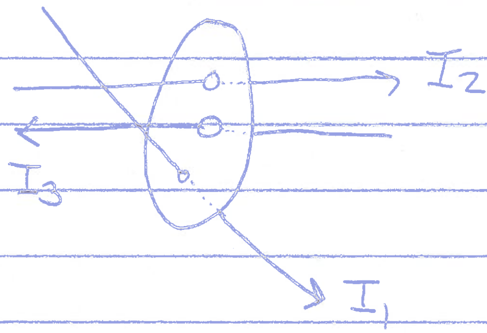
- integral of $\vec{B}$ along a circular path at distance s from the wire

$$\oint \vec{B}\cdot d\vec{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \frac{\mu_0 I}{2\pi s}\oint dl = \mu_0 I$$

circle

- turns out any loop enclosing the wire gives $\mu_0 I$

- Bundle of wires passing through the loop



$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$, for volume current densities

$$I_{enc} = \int \vec{J}\cdot d\vec{a} \rightarrow \text{flux of current density}$$

- Stokes' theorem:

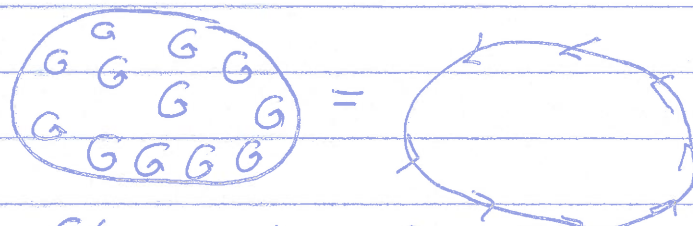
$$\oint \vec{B}\cdot d\vec{l} = \int (\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \mu_0 I_{enc} = \mu_0 \int \vec{J}\cdot d\vec{a}$$

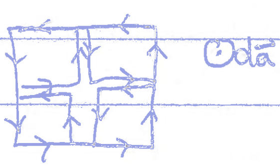
$$\Rightarrow \boxed{\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}} \rightarrow \text{Ampère's law (mag. statics)}$$

- constructed from infinite straight wires
  - cannot always do this

- Stoke's Theorem:

- tiny swirls add up to swirl at boundary





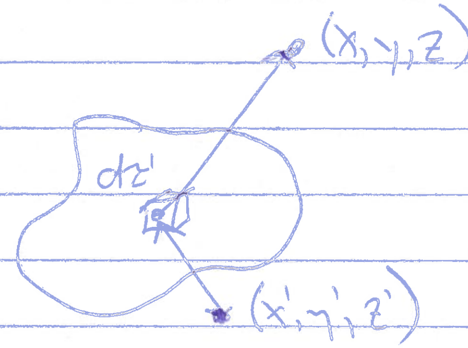
$$\int_S (\nabla\times\vec{v})\cdot d\vec{a} = \oint_{\partial S} \vec{v}\cdot d\vec{l}$$

- Biot-Savart law for current density, $\vec{J}$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

$$\vec{B} \rightarrow \vec{B}(x,y,z)$$
$$\vec{J} \rightarrow \vec{J}(x',y',z')$$
$$\vec{\mathscr{r}} = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$$
$$d\tau' = dx'dy'dz'$$

- integrate over primed coordinates
- $\nabla\cdot\vec{B}$ + $\nabla\times\vec{B}$ over unprimed coordinates



$$\nabla\cdot\vec{B} = \frac{\mu_0}{4\pi}\int \nabla\cdot\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) d\tau'$$

prod. rule 6

$$\nabla\cdot\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot(\nabla\times\vec{J}) - \vec{J}\left(\nabla\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$$

$\nabla\times\vec{J} = 0$, $\vec{J} \rightarrow \vec{J}(x',y',z')$, vary $(x,y,z) \rightarrow \vec{J}$ does not change

$-(\vec{J}\cdot\nabla)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = (\vec{J}\cdot\nabla')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$

$* \ \frac{\partial}{\partial x} f(x-x') = -\frac{\partial}{\partial x'} f(x-x')$

$(\vec{J}\cdot\nabla')\left(\frac{x-x'}{\mathscr{r}^3}\right) = \nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\vec{J}\right] - \left(\frac{x-x'}{\mathscr{r}^3}\right)\underbrace{(\nabla'\cdot\vec{J})}_{0}$

$\Rightarrow \int_{vol} \nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\vec{J}\right] d\tau' = \underbrace{\oint \frac{(x-x')}{\mathscr{r}^3}\vec{J}\cdot d\vec{a}}_{0}$

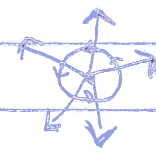
$\left(\nabla\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 0 \quad , \quad \vec{\mathscr{r}} =$

-point charge at origin $\vec{r}' = 0$

$\int_a^b \frac{1}{r^2}\,dr = -\frac{1}{r}$

$a = b \rightarrow \oint\left(\frac{\hat{r}}{r^2}\right)\cdot d\vec{r}$ $\left(\frac{}{^2}\right)\cdot d\vec{a}$

$\Rightarrow \nabla\times\frac{\hat{r}}{r^2} = 0$



$\oint \times\left(\frac{\hat{r}}{r^2}\right) dl = 0$

$\Rightarrow \boxed{\nabla\cdot\vec{B} = 0} \Rightarrow$ magnetic charges don't exist

Curl of $\vec{B} \rightarrow$

$$\nabla\times\vec{B} = \frac{\mu_0}{4\pi}\int \nabla\times\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'$$

-product rule 8 (4-terms) $\Rightarrow$

-derivatives of $\vec{J}$ dropped.

$$\nabla\times\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \vec{J}\underbrace{\left(\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)} - \underbrace{\cancel{(\vec{J}\cdot\nabla)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}}}_{0}$$

-page 233

$\nabla'\cdot\vec{J} = 0 \rightarrow$ steady

$\oint_S \vec{J}\cdot d\vec{a} = 0$

$$\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\mathscr{r})$$

$$\Rightarrow \nabla\times\vec{B} = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\,4\pi\,\delta^3(\vec{r}-\vec{r}')\,d\tau' = \mu_0\vec{J}(\vec{r})$$

$\int \delta^3(\vec{r})\,d\tau = 1$

1D: $\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$

$\underbrace{\int}_{\text{all space}} f(\vec{r})\,\delta^3(\vec{r}-\vec{a})\,d\tau = f(\vec{a})$

- <u>Ampère's law</u>: holds for all $\bar{J}$

$$\boxed{\nabla \times \vec{B} = \mu_0 \bar{J}} \rightarrow \text{differential version}$$

$\rightarrow$ Stoke's theorem $\int (\nabla \times \vec{B}) \cdot d\vec{a} = \oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{a}$

$= \mu_0 I_{enc}$

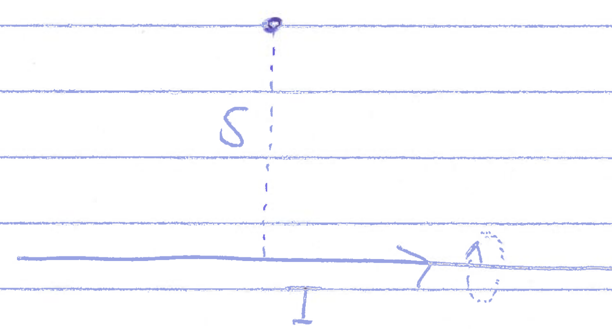
$$\boxed{\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}} \rightarrow \text{integral version}$$

loop $\rightarrow$ "Amperian loop" encloses current

Electrostatics : Coulomb's law $\rightarrow$ Gauss

Magnetostatics : Biot-Savart law $\rightarrow$ Ampère

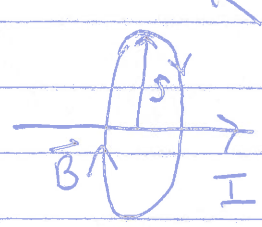
<u>Ex</u>: 5.7



$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$, $I_{enc} = I \Rightarrow \vec{B} \cdot d\vec{l} = B dl$

$B \oint dl = \mu_0 I \Rightarrow B \cdot 2\pi \cdot s = \mu_0 I$

path is circle

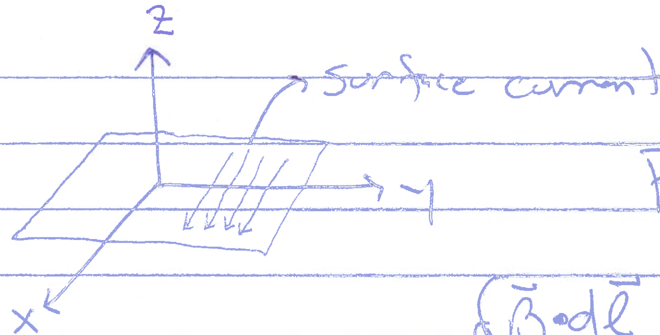


$$\Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}$$

Ex: 5.8

$\vec{K} = \sigma \vec{v}$, $\vec{K} = \frac{dI}{dl_\perp}$

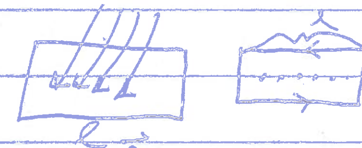
$\vec{B}$



$\vec{K} = K\hat{x} \rightarrow$ uniform

$\oint \vec{B}\cdot d\vec{l}$

$\leftarrow \vec{B} : z > 0$



$\vec{B} \rightarrow : z < 0$

$$\int_{above} \vec{B}\cdot d\vec{l} = Bl \quad , \int_{below} \vec{B}\cdot d\vec{l} = Bl \quad , \text{ sides } \vec{B}\cdot d\vec{l} = 0$$

End. Mon. Sept. 22

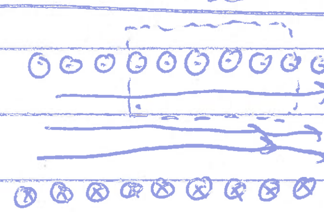
$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} = 2Bl \; ; \; I_{enc} = K\cdot l$$

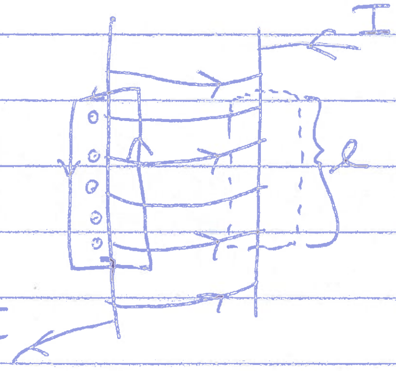
$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$

$$\mu_0 K l = 2Bl \Rightarrow B = \begin{cases} \left(\frac{\mu_0}{2}\right) K \hat{y} \; ; \; z < 0 \\ -\left(\frac{\mu_0}{2}\right) K \hat{y} \; ; \; z > 0 \end{cases}$$

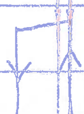
Ex:

$n = \frac{\text{turns}}{\text{length}}$

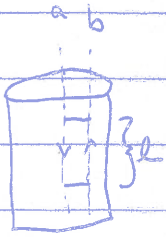




$\uparrow B$

- center:  $\int(-B)\cdot dl + \int B(dl)$

$I_{enc} = 0$



$[B(a) - B(b)]\,l = 0$

$B(a) = B(b)$

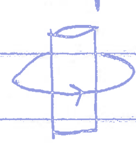
$\oint \vec{B}\cdot d\vec{l} = Bl \rightarrow$ B field

$= \mu_0 n l I$ inside

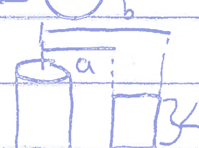
$\Rightarrow \vec{B} = \mu_0 n I \hat{z}$ solenoid

0, outside only

- ampere loop outside $= 0$.



$I_{enc} = 0$



$\oint \vec{B}\cdot d\vec{l} = [B(a) - B(b)]\,l = \mu_0 I_{enc} = 0$

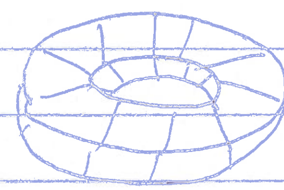
$B(a) = B(b)$

$\rightarrow 0$ $B \rightarrow 0 @ \infty$

Ampére's law is useful for 4 cases:

- Infinite lines $\qquad \vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$
- Infinite planes $\qquad \rightarrow$ uniform $\vec{K}$: $\sim \left(\frac{\mu_0}{2}\right) K$
- Infinite solenoids $\qquad \vec{B} = \mu_0 n I \hat{z}$
- Toroids

Ex: 5.10 → toroid

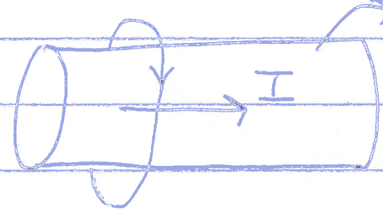


$$\Rightarrow \vec{B}(\vec{r}) = \begin{cases} \frac{\mu_0 N I}{2\pi s}\hat{\phi}, & \text{inside} \\ 0, & \text{outside} \end{cases}$$

5.14: pg. 239

(a)

surface only, uniform



- $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \Rightarrow B \cdot 2\pi s = \mu_0 I$

$$\Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}, \quad s > a$$

$$\vec{B} = 0, \quad s < a$$

- why?

- no current inside

(b) Current is distributed so $\vec{J} = ks$

$$J = ks \;;\; I = \int_0^a ks\,da = \int_0^a ks(2\pi s)\,ds$$

$$= 2\pi k a^3/3 \Rightarrow k = \frac{3I}{2\pi a^3} \qquad \frac{2\pi}{3}\left(\frac{3I}{2\pi a^3}\right)s^3$$

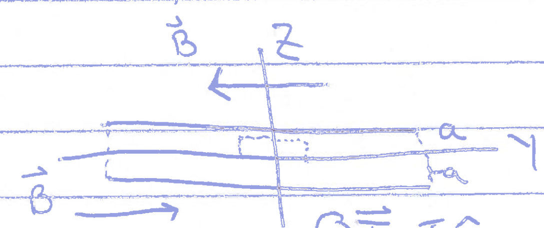
$$I_{enc} = \int_0^s ks' 2\pi s'\,ds' = \frac{2\pi}{3}ks^3 = \frac{Is^3}{a^3}, \; s<a$$

$s>a$: $I_{enc} = I \Rightarrow \frac{\mu_0 I}{2\pi s}\hat\phi = \vec B$

$s<a$: $\frac{\mu_0 I s^2}{2\pi a^3}\hat\phi = \vec B$

$$\boxed{\begin{aligned}\nabla\times\vec B &= \frac{1}{s}\left[\frac{\partial}{\partial s} s\left(\frac{\mu_0 I s^2}{2\pi a^2}\right) - \underbrace{\frac{\partial}{\partial \phi}\cancel{\left(\frac{\mu_0 I s^3}{2\pi a^2}\right)}}_{0}\right]\hat z \\ &= \frac{1}{s}\left[\frac{3s^2\mu_0 I}{2\pi a^2}\right]\hat z = \frac{3\mu_0 I s}{2\pi a^2}\hat z \\ &= \mu_0\left(\frac{3I}{2\pi a^2}\right)s\hat z = \mu_0 k s\hat z = \mu_0\vec J\end{aligned}}$$

Prob. 5.15: thick slab from $z=-a$ to $+a$ has
pg. 240 current $\vec J = J\hat x$



$\vec B(z=0)=0$, $B\cdot l = \mu_0 J\cdot \underbrace{l\cdot z}_{area} \Rightarrow \vec B = J\mu_0 z(-\hat y)$

$(-a<z<a)$

- symmetry
- opposite directions

$z>a \rightarrow I_{enc} = \mu_0 l a J$

$\Rightarrow \vec B = -\mu_0 J a\hat y \;;\; z>a$

$\vec B = \mu_0 J a\hat y \;;\; z<a$

$\vec{B} = -\mu_0 J z \hat{y}$

$$\nabla \times \vec{B} = \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) \hat{x} + \dots$$

$$= -\frac{\partial B}{\partial z} \hat{x} = \frac{-\partial}{\partial z} \left(\mu_0 J z \right) \hat{x} = \mu_0 J \hat{x}$$

- real world problem

Prob 5.20: 242

(a) density of mobile charges ρ, 1 atom $\to$ 1 e^- assumption

$$\rho = \frac{\text{Charge}}{\text{vol}} = \frac{\text{Charge}}{\text{atom}} \times \frac{\text{atoms}}{\text{mol}} \times \frac{\text{mol}}{\text{gram}} \times \frac{\text{gram}}{\text{vol}} \qquad M \equiv \text{molar mass}$$

$$= (e) \times N \times \frac{1}{M} \times \text{density}(Cu)$$

$N \uparrow$ Avogadro's #, M atomic mass Cu

1C = 1A × 1s

0.1 A → lethal

100 mA - 200 mA ventricular fibrilation

$e = \cancel{1.4 \times 10^4 \, C/cm}$ $1.6 \times 10^{-19} C$

$N = 6 \times 10^{23}$

$M = 64 g/mol$

$d = 9.0 g/cm^3$

$\rho = 1.4 \times 10^4 \frac{C}{cm^3}$

14,000 C

(b) average e^- velocity in 1mm diameter wire 1A

$$J = \frac{I}{\pi s^2} = \rho v \Rightarrow v = \frac{I}{\pi s^2 \rho} = \frac{1}{\pi (2.5 \times 10^{-3})(1.4 \times 10^4)}$$

$= 9.1 \times 10^{-3} cm/s$, $33 cm/hr.$

(c) Force between 2 wires 1 cm apart?

$$f_m = \frac{\mu_0}{2\pi}\left(\frac{I^2}{d}\right) = \frac{4\pi\times10^{-7}}{2\pi} = 2\times10^{-7}\ \frac{N}{cm}$$

$\leftarrow$ length along wire

(d) * Magnetic Vector Potential:

- Electrostatics $\rightarrow$ $\nabla\times\vec{E}=0 \Rightarrow \vec{E}=-\nabla V$

- $\nabla\cdot\vec{B}=0 \Rightarrow \vec{B}=\nabla\times\vec{A}$ $\rightarrow$ section 1.6.2

- $\vec{A}$ is a vector field

$\nabla\cdot(\nabla\times\vec{A})=0$, diverg. of curl is always 0

$$\nabla\times\vec{B} = \nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0 J$$

- you can add to $\vec{A}$ any function whose curl vanishes w/out affecting $\vec{B}$

$$\vec{B} = \nabla\times(\vec{A}+\vec{C})$$
$$= \nabla\times\vec{A} + \cancel{\nabla\times\vec{C}}\ 0$$

- if $\nabla\times\vec{C}=0$, then $\vec{C}=\nabla F$, F is scalar

Friday
Sept. 26th

- $\vec{E}$-fields generated by stationary charges

$\nabla \times \vec{E} = 0$ — potential energy per charge

- immediately allows for us to write $\vec{E} = -\nabla V$

Poisson's eq.

$$\nabla \times (-\nabla V) = 0, \quad \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} = \nabla \cdot (-\nabla V) \Rightarrow \boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}}$$

- irrotational fields always can be written as a gradient of some scalar.
- obviously simplifies ~~things~~ electrostatics since V is a scalar vs. vector

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')\hat{\eta}\,d\tau'}{\eta^2} \; ; \; V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')\,d\tau'}{\eta}$$

- steady current $\vec{B}$-fields + all fields

$$\nabla \cdot \vec{B} = 0$$, div. field → no end points to lines

$$\Rightarrow \vec{B} = \nabla \times \vec{A}, \quad \vec{A} \equiv \text{vector potential}$$

$\nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A})$; divergence of curl always zero

→ "solenoidal" fields can always be written this way

→ not obvious why this is useful at first since one vector field is replaced by another

→ curl of $\vec{A}$ gives $\vec{B}$-field by $\underline{\nabla \cdot \vec{A}}$ has no significance physically

$V = -\int_0^{\vec{r}} \vec{E}\cdot d\vec{l}$

$V'(\vec{r}) = K + V(\vec{r})$

- therefore we can choose $\nabla\cdot\vec{A}$ to be whatever we want
- $\vec{A}$ is invariant under transformation

$$\vec{A} \Longrightarrow \vec{A}_0 + \nabla\psi(\vec{r}), \quad \vec{A}_0 \text{ is not divergenceless}$$

- changing reference points for elec. potential

$$V'(\vec{r}) = -\int_{O'}^{\vec{r}} \vec{E}\cdot d\vec{l} = \int_{O'}^{O} \vec{E}\cdot d\vec{l} - \int_{O}^{\vec{r}} \vec{E}\cdot d\vec{l} = K + V(\vec{r})$$

$\Rightarrow$ potential difference between 2 points $\vec{a}, \vec{b}$

$$\left.\begin{aligned} V'(\vec{b}) &= K + V(\vec{b}) \\ V'(\vec{a}) &= K + V(\vec{a}) \end{aligned}\right\} \quad V \rightarrow V + K$$

$$V'(\vec{b}) - V'(\vec{a}) = V(\vec{b}) - V(\vec{a}) \rightarrow \text{constant cancels}$$

- $V + K$ & V give same $\vec{E}$ field

$$\underbrace{-\nabla(V + K) = -\nabla V}_{=\vec{E}}, \quad \nabla K = 0$$

- in analogy to magnetism,

2nd
Wed.
24th

- BC
- multipole exp.

Magnetic Vector Potential:

- div. → lines end

- curl → no end

- electrostatics $\nabla \times \vec{E} = 0 \Rightarrow \vec{E} = -\nabla V$

- $\nabla \cdot \vec{B} = 0 \Rightarrow \vec{B} = \nabla \times \vec{A}$ (section 1.6.2)

$$\nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) = 0, \text{ div. of curl } \underline{\text{always}}\ 0$$

here

- One can add any function whose curl vanishes to $\vec{A}$
  - w/ no effect on $\vec{B}$

  we can add
  - gradient of a scalar $\rightarrow \nabla \Psi(\vec{r})$ curl vanishes

  - $\nabla \times (\nabla \Psi(\vec{r})) = 0$, for $\underline{\text{all}}$ $\Psi(\vec{r})$

- since we can add any function $\nabla \Psi(\vec{r})$ to $\vec{A}$, it is always possible to make div. vanish (most natural) → curl vanishes

$$\boxed{\nabla \cdot \vec{A} = 0} \rightarrow \text{Coulomb gauge}$$

original potential

- assume $\vec{A} = \vec{A}_0 + \nabla \Psi(\vec{r})$, $\vec{A}_0$ is not $\underline{\text{divergenceless}}$

"Gauge transformation"

$$\nabla \cdot \vec{A} = \nabla \cdot \vec{A}_0 + \nabla^2 \Psi(\vec{r})$$

- if we let $\nabla \cdot \vec{A} = 0$, is it possible to find $\Psi(\vec{r})$?

$$\nabla^2 \Psi(\vec{r}) = -\nabla \cdot \vec{A}_0 \rightarrow \text{Poisson's equation.}$$

$$\Psi(\vec{r}) = \frac{1}{4\pi} \int \frac{\nabla \cdot \vec{A}_0 \, d\tau'}{r}, \text{ if } \nabla \cdot \vec{A}_0 \rightarrow 0 \text{ @ infinity}$$

- We can always find a $\Psi(\vec{r})$ of this form to force $\nabla\cdot\vec{A}=0$, or we can choose to make $\nabla\cdot\vec{A}=0$
- we are only restricted to the curl $\nabla\times\vec{A}=\vec{B}$

$\nabla\cdot\vec{A}=0$, best for mag. stat.

$$\nabla\times\vec{B}=\nabla\times(\nabla\times\vec{A})=\nabla(\cancel{\nabla\cdot\vec{A}}^{\,0})-\nabla^2\vec{A}=\mu_0\vec{J}$$

- if we want, we can choose $\nabla\cdot\vec{A}$ to be anything → Lorentz:

$$\boxed{\nabla\cdot\vec{A}=\mu_0\epsilon_0\frac{\partial V}{\partial t}}$$

→ Ampere's law

$$\Rightarrow \boxed{\nabla^2\vec{A}=-\mu_0\vec{J}}$$, Poisson's eq. in 3D

$$\boxed{\nabla^2 V=-\rho/\epsilon_0 \qquad V=\frac{1}{4\pi\epsilon_0}\int\frac{\rho}{\mathscr{r}}d\tau'}$$

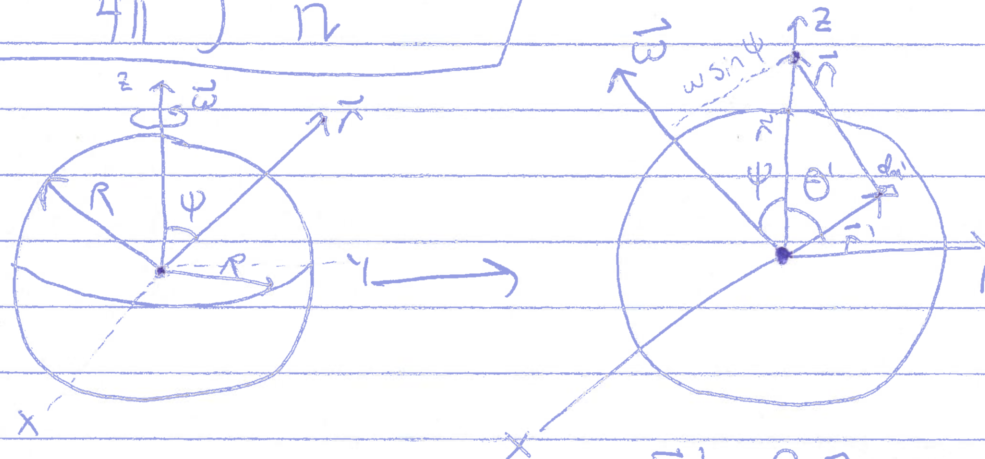
- Solution for $\vec{J}\to 0$ @ ∞ is

- front of book

$$\boxed{\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}\,')\,d\tau'}{\mathscr{r}}}$$

<u>Ex: 5.11</u>

- Shell, w/ σ



$\vec{r}\,'=R\hat{r}$

→ switch to $\vec{r}=r\hat{z}$

- $\vec{\omega}$ in yz plane

$$\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int\frac{\vec{K}(\vec{r}\,')\,da'}{\mathscr{r}} \quad , \vec{K}=\sigma\vec{v}$$

$$\mathscr{r}=\sqrt{R^2+r^2-2Rr\cos\theta'}$$
- law of cosines

$$da'=R^2\sin\theta'\,d\theta'\,d\phi'$$

$$\vec{V} = \vec{\omega} \times \vec{r}\,' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega \sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

$$= R\omega\Big[-(\cos\psi \sin\theta' \cancel{\sin\phi'})\hat{x}\ (\cos\psi \sin\theta' \cancel{\cos\phi'} \underbrace{- \sin\psi\cos\theta'})\hat{y}$$

$$+ (\sin\psi \sin\theta' \cancel{\sin\phi'})\hat{z}\Big]$$

$$\int_0^{2\pi} \sin\phi'\, d\phi' = \int_0^{2\pi} \cos\phi'\, d\phi' = 0$$

$$\vec{A}(\vec{r}) = -\frac{\mu_0 R^3 \sigma \omega \sin\psi}{2} \int_0^{\pi} \frac{\cos\theta' \sin\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}}\, \hat{y}$$

$$u \equiv \cos\theta'$$

$$\int_{-1}^{+1} \frac{u}{\sqrt{R^2+r^2-2Rru}} = -\frac{(R^2+r^2+Rru)\sqrt{R^2+r^2-2Rru}}{3R^2r^2}\Bigg|_{-1}^{+1}$$

$$= -\frac{1}{3R^2r^2}\Big[(R^2+r^2+Rr)\,|R-r| - (R^2+r^2-Rr)(R+r)\Big]$$

$$(r-R) \to |R-r|$$

$(R-r) \to |R-r|$: $R > r \to \dfrac{2r}{3R^2}$, $r < R \to \dfrac{2R}{3r^2}$

$$\vec{\omega}\times\vec{r} = -\omega r \sin\psi \qquad \vec{A}(\vec{r}) = \begin{cases} \dfrac{\mu_0 R \sigma}{3} (\vec{\omega}\times\vec{r}) & \text{in} \\ \dfrac{\mu_0 R^4 \sigma}{3r^3} (\vec{\omega}\times\vec{r}) & \text{out} \end{cases}$$

$\longrightarrow$ 2 pages

ex. 5.11 → contin.

- revert to natural coordinates → $\vec{\omega} = \omega \hat{z}$

$$\vec{A}(r,\theta,\phi) = \begin{cases} \dfrac{\mu_0 R \omega \sigma}{3} r \sin\theta \ \hat{\phi} & r \leq R \\ \dfrac{\mu_0 R^4 \omega \sigma}{3} \dfrac{\sin\theta}{r^2} \ \hat{\phi} & r \geq R \end{cases}$$

$$\vec{B} = \nabla \times \vec{A} = \frac{2\mu_0 R \omega \sigma}{3} (\cos\theta \ \hat{r} - \sin\theta \hat{\theta})$$

$$= \frac{2}{3} \mu_0 \sigma R \omega \hat{z} = \frac{2}{3} \mu_0 \sigma R \vec{\omega}$$ → inside shell → uniform

- $(\cos\theta \hat{r} - \sin\theta \hat{\theta})$

$$\cos\theta \ \hat{r} = \sin\theta\cos\phi\cos\theta \ \hat{x} + \sin\theta \sin\phi \cos\theta \ \hat{y} + \cos^2\theta \ \hat{z}$$

$$-\sin\theta \ \hat{\theta} = -\sin\theta\cos\theta\cos\phi \ \hat{x} - \cos\theta \sin\theta \sin\phi \ \hat{y} + \sin^2\theta \ \hat{z}$$

$$= (\cos^2\theta + \sin^2\theta)\hat{z}$$

- important → ϕ is in same direction as current

Monday
Sept. 29th
begin

- Q.M. → potentials only
- QFT quantum field theory → handle only potentials

- In-class quiz
  Friday → section
  5.3.3
  Ampère's law
- Exam next Friday 10th
  Ch. 5 + → however far
  in 6 we get

- Magnetic V

$\nabla \cdot \vec{B}$ ... , $\nabla \cdot (\nabla \times \vec{A}) = 0$ ✓

$\nabla \times \vec{B} =$ ... $- \nabla^2 \vec{A} = \mu_0 \vec{J}$

• choose ∇ ... $\boxed{\;\ldots = \mu_0 \vec{J}}$
(mag. statics)

→ if $\vec{J} \to 0$ @ infinity, $\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')\, d\tau'}{\mathcal{r}}}$

- units of $\vec{A}$ → momentum per charge

$1C = 1A \cdot s$

$$\frac{Vs}{m} = \left(\frac{kg\, m^2}{A s^3}\right) \cdot \frac{s}{m} = \frac{kg\, m}{A s^2} = \left(\frac{kg\, m}{s}\right) \underbrace{\frac{1}{A \cdot s}}_{1C} ;$$

B-field : $T \to \frac{N}{A \cdot m} = \left(\frac{kg\, m}{s^2}\right) \cdot \frac{s}{C} \cdot \frac{1}{m} = \frac{kg}{C \cdot s}$

$\vec{B} = \nabla \times \vec{A}$, $\vec{B} = \frac{\text{units of } \vec{A}}{m\,(\text{distance})}$ → (units of $\vec{B}$) · distance → units of $\vec{A}$

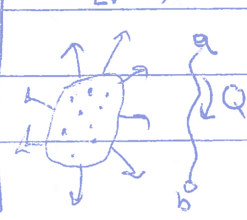
$\underbrace{\left(\frac{kg}{C \cdot s}\right)}_{T} \cdot m \to \frac{kg\, m}{s} \frac{1}{C} \to \frac{\text{momentum}}{\text{charge}}$

elec. st
$W = \int_a^b \vec{F} \cdot d\vec{l} = Q \int_a^b \vec{E} \cdot d\vec{l}$
$= Q[V(\vec{b}) - V(\vec{a})] = \frac{W}{Q}$

- only useful in certain contexts
- moving charge has momentum

point charge - Lagrangian : $\mathcal{L} = \frac{1}{2} m v^2 - \cancel{V} \overset{\text{P.E.}}{=} \frac{1}{2} m v^2 - q\,(V - \vec{v} \cdot \vec{A})$

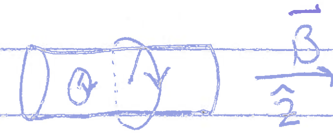
$q\,(V - \vec{v} \cdot \vec{A})$ → generalized P.E.



Ex: 5.12 $\vec{A}(\vec{r})$ of infinite solenoid, n turns/length
radius R, current I

→ I goes to infinity → so cannot use

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\, dl'}{\eta}$$



why do this?

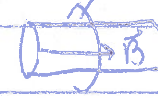
$\nabla\times\vec{A} = \vec{B}$

stoke's

- trick: $\oint \vec{A}\cdot d\vec{l} = \int (\nabla\times\vec{A})\cdot d\vec{a} = \int \vec{B}\cdot d\vec{a} = \Phi_B$

flux through loop

$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$, $\vec{B} \to \vec{A}$ + $\mu_0 I_{enc} \to \Phi$



- for a circle

$$\oint \vec{A}\cdot d\vec{l} = A(2\pi s) = \int \vec{B}\cdot d\vec{a} = \mu_0 n I(\pi s^2)$$

(we knew $\vec{B}$!)

$$\vec{A} = \frac{\mu_0 n I}{2} s\hat{\phi} \quad , \quad s<R$$

s inside solenoid

$\int \vec{B}\cdot d\vec{a} = \mu_0 n I(\pi R^2)$ ← outside $s \geq R$

$$= A(2\pi s) \Rightarrow \vec{A} = \frac{\mu_0 n I}{2}\frac{R^2}{s}\hat{\phi} \quad , \quad s \geq R$$

$$-\ \nabla\times\vec{A} = \dots + \frac{1}{s}\left[\frac{\partial}{\partial s}(sA_\phi) - \frac{\partial}{\partial\phi}\overset{0}{\cancel{(A_s)}}\right]\hat{z}$$

$s > R$:

$$\nabla \times \vec{A} = \frac{1}{s}\left[\frac{\partial}{\partial s}\left(s\,\frac{\mu_0 n I}{2}\,\frac{R^2}{s}\right)\right]\hat{z} = 0 \quad ✓$$

$s < R$: $$\nabla \times \vec{A} = \frac{1}{s}\left[\frac{\partial}{\partial s}\left(s\,\frac{\mu_0 n I s}{2}\right)\right]\hat{z} = \frac{1}{s}\left[\mu_0 n I s\right]$$

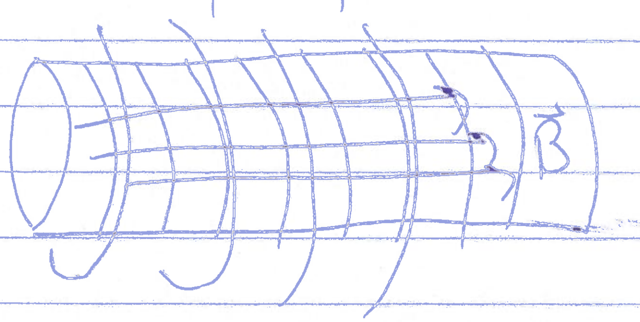
$$= \mu_0 n I\,\hat{z} \quad ✓$$

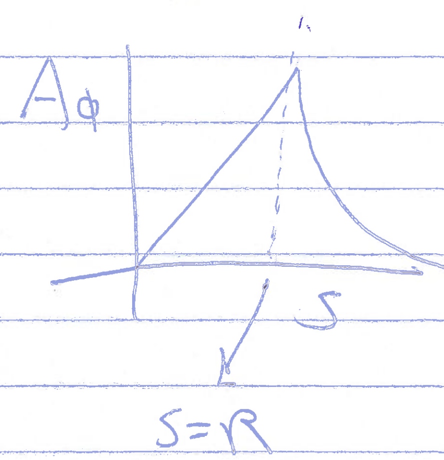
$$\nabla \cdot \vec{A} = \underbrace{\cancel{\frac{1}{s}\frac{\partial}{\partial s}\left(sA_s\right)}}_{0} + \underbrace{\cancel{\frac{1}{s}\frac{\partial}{\partial \phi}A_\phi}}_{0} + \underbrace{\cancel{\frac{\partial A_z}{\partial z}}}_{0}$$

$$\cancel{s \geq R:\ \frac{1}{s}\frac{\partial}{\partial s}\left(\frac{\mu_0 n I}{2}\,\frac{R^2}{s}\right) =} \qquad \nabla \cdot \vec{A} = 0 \quad ✓$$

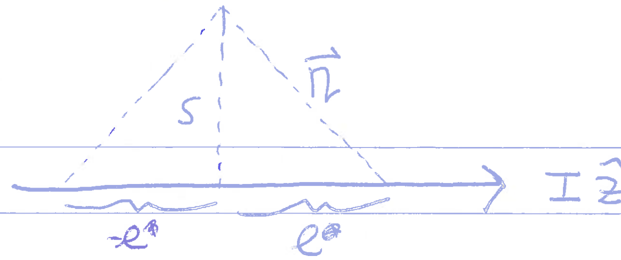
$$\cancel{s < R:\ \frac{1}{s}\frac{\partial}{\partial s}\left(\frac{\mu_0 n I s}{2}\right) = \frac{\mu_0 n I}{2s}}$$

$\rightarrow$ $\vec{A}$ is typically the direction of current





- finite line segment



$$|\vec{r}| = \sqrt{s^2+z^2}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_{-l}^{+l} \frac{d\vec{l}'}{r} = \frac{\mu_0 I}{4\pi}\hat{z}\int_{-l}^{+l}\frac{dz}{\sqrt{s^2+z^2}}$$

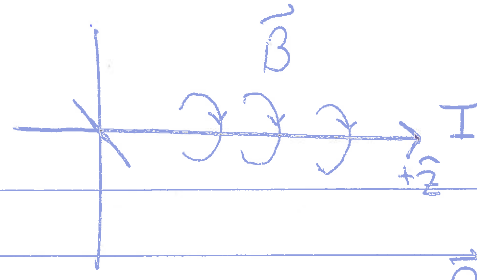
$$= \frac{\mu_0 I}{4\pi}\ln\left[\frac{l+(l^2+s^2)^{1/2}}{-l+(l^2+s^2)^{1/2}}\right]\hat{z},\ l\to\infty,\ \text{expand}$$

$$s = r\cos\theta \quad \frac{1}{r} = \frac{\cos\theta}{s} \quad dl' = \frac{s}{\cos^2\theta}\,d\theta$$

$$\int \frac{s}{\cos^2\theta}\cdot\frac{\cos\theta}{s}\,d\theta$$

$$= \int \frac{1}{\cos\theta}\,d\theta$$

$$\rightarrow \boxed{-\frac{\mu_0 I}{4\pi}\ln(s) = \vec{A}}$$



$\vec{B} \to \odot$ $s>0$ above

$\vec{B} \to \otimes$ $s<0$ below

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

$$\vec{B} = \nabla \times \vec{A} = \left[\frac{1}{s}\frac{\partial A_{\cancel{s}z}}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{s} \qquad \vec{I} = I\hat{z}$$

$$+ \left[\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right]\hat{\phi}$$

$$+ \frac{1}{s}\left[\frac{\partial}{\partial s}(sA_\phi) - \frac{\partial A_s}{\partial \phi}\right]\hat{z} \quad = \vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

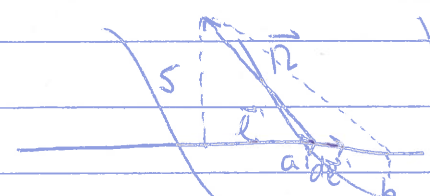
$$\vec{A} = (\overset{0}{A_s}, \overset{0}{A_\phi}, A_z) \longrightarrow A = A_z\hat{z}$$

$$\frac{\mu_0 I}{2\pi s} = \frac{\partial A_s}{\partial z} - \underbrace{\frac{\partial A_z}{\partial s}} \to \text{only } \hat{z} \text{ direction}$$

$$\frac{\mu_0 I}{2\pi s} = -\frac{\partial A_z}{\partial s} \Rightarrow -\int \frac{\mu_0 I}{2\pi s} ds = A_z + \text{const.}$$

$$= -\frac{\mu_0 I}{2\pi}\ln(s) = A_z \Rightarrow \vec{A} = -\frac{\mu_0 I}{2\pi}\ln(s)\hat{z}$$

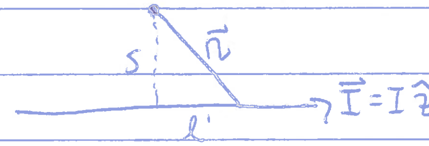
-directly:



$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_a^b \frac{dl'}{\mathscr{r}}$$

$$= \frac{\mu_0 I}{4\pi}\int_a \frac{dl}{\sqrt{s^2+l'^2}}$$

-next page:



end Mon.
Sept. 29

$$\nabla \times \vec{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{s} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\underline{\frac{\partial (s v_\phi)}{\partial s}} - \frac{\partial v_s}{\partial \phi}\right]\hat{z}$$

5.24: what current density gives $\boxed{\vec{A} = k\hat{\phi}}$

$$\vec{B} = \nabla \times \vec{A} = \frac{1}{s}\frac{\partial (sk)}{\partial s}\hat{z} = \boxed{\frac{k}{s}\hat{z} = \vec{B}}$$

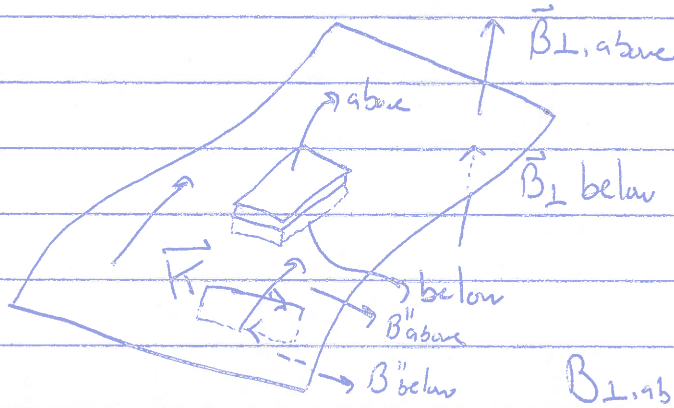
$$\nabla \times \vec{B} = \mu_0 \vec{J} =$$

$$\Rightarrow \vec{J} = \frac{1}{\mu_0}\left[-\frac{\partial}{\partial s}\left(\frac{k}{s}\right)\right]\hat{\phi} = \boxed{\frac{k}{\mu_0 s^2}\hat{\phi} = \vec{J}}$$

→ field $\vec{A}$ of wire

Boundary Conditions:

- B-field is discontinuous at a surface current



$\int \nabla\cdot\vec{B}\, d\tau = \oint \vec{B}\cdot d\vec{a} = 0$
div. theorem

$\oint \vec{B}\cdot d\vec{l} = 0 \rightarrow$
parallel to $\vec{K}$

$B_{\perp,ab.}\cdot Area = B_{\perp,bel.}\cdot Area$

$$\nabla\cdot\vec{B} = 0 \Rightarrow \oint \vec{B}\cdot d\vec{a} = 0 \Rightarrow B_{\perp,above} = B_{\perp,below}$$

– continuous across surface

$$\oint \vec{B}\cdot d\vec{l} = \left(B''_{above} - B''_{below}\right)l = \mu_0 I_{enc} = \mu_0 K l$$

$$B''_{above} - B''_{below} = \mu_0 K$$

- discont. is perp. to the normal $\hat{n}$ + current direction
surface

$\Rightarrow \vec{B}_{above} - \vec{B}_{below} = \mu_0 (\vec{K} \times \hat{n})$

$\rightarrow \vec{A}_{above} = \vec{A}_{below}$, $\nabla \cdot \vec{A} = 0$ guarantee's that the normal component is continuus

$$\nabla \times \vec{A} = \vec{B} \Rightarrow \oint \vec{A} \cdot d\vec{l} = \int (\nabla \times \vec{A}) \cdot d\vec{a} = \int \vec{B} \cdot d\vec{a} = \Phi_{mag}$$

$\rightarrow$ as thickness goes to 0 for loop $\rightarrow \Phi_{mag} \rightarrow 0$

$$(\vec{A}_{above} - A_{below})\, l = 0 \Rightarrow \vec{A}_{above} = \vec{A}_{below}$$

$$\frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{K}$$

Ex $\quad \vec{K} = K\hat{x}$

$$\vec{B} = \begin{cases} \frac{\mu_0 K}{2}\hat{y} & ; z < 0 \\ -\frac{\mu_0 K}{2}\hat{y} & ; z > 0 \end{cases}$$

$$\vec{B}_{above} - \vec{B}_{below} = \mu_0 (\vec{K} \times \hat{n})$$

$$= -\frac{\mu_0 K \hat{y}}{2} - \frac{\mu_0 K \hat{y}}{2} = -\mu_0 K \hat{y}$$

$$\nabla \times \vec{A} = \dots \hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_y}{\partial x}\right)\hat{y} + \dots \hat{z} = \vec{B}$$

$$\frac{\partial A_x}{\partial z} - \cancel{\frac{\partial A_y}{\partial x}}^{\,0} = -\frac{\mu_0 K}{2}\hat{y} ; \text{ above } {}^{z>0}$$

$$\frac{\partial A_x}{\partial z} = \frac{\partial A_x}{\partial n} = -\frac{\mu_0 K}{2}\hat{y}, \; z > 0$$

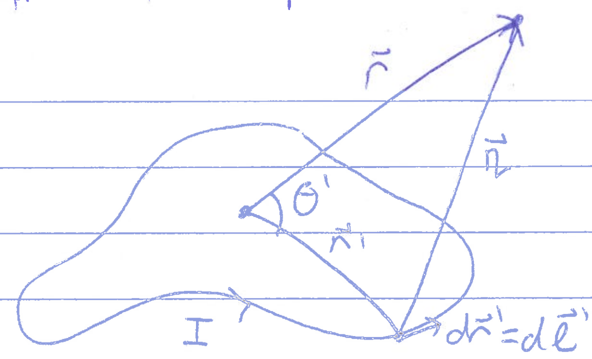
$$\frac{\partial A_x}{\partial z} = \frac{\mu_0 K}{2}\hat{y}, \; z < 0$$

$$\rightarrow \frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{K}$$

$$\vec{K} \times \hat{n} \quad \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ K & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\hat{y}K$$

$$\Rightarrow \mu_0 (\vec{K} \times \hat{n}) = -\mu_0 K \hat{y} \checkmark$$

- Multipole expansion for vector potential
- approx. formula for vec. pot. at a distant point, write pot. in power series



$$\eta^2 = r^2 + r'^2 - 2rr'\cos\theta' = r^2\Big[1 + \underbrace{\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta'}_{\epsilon}\Big]$$

$$\eta = r\sqrt{1+\epsilon}\,, \quad \epsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right)$$

- binomial expansion; $\dfrac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3$

$$\frac{1}{\eta} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \ldots\right)$$

$$\frac{1}{\eta} = \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\theta'\right)^2\right.$$

$$\vdots$$

$$\frac{1}{\eta} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta') \rightarrow \text{Legendre Polynomials}$$

- Solution of Laplace's eq. in spherical units $\nabla^2\phi = 0$

$$\vec{A}(\vec{r}) = \cancel{\frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\vec{l}'\right] = \mu_0}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{1}{\eta}\,d\vec{l}' = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\theta')\,d\vec{l}'$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\left[\underbrace{\frac{1}{r}\oint d\vec{l}'}_{\text{monopole}} + \underbrace{\frac{1}{r^2}\oint r'\cos\theta' d\vec{l}'}_{\text{dipole}} + \dots\right]$$

$$\oint d\vec{l}' = 0$$

- vector potential assumes $\nabla\cdot\vec{B} = 0$, so it makes sense that this is consistent

$$\vec{A}_{dip} = \frac{\mu_0 I}{4\pi r^2}\oint r'\cos\theta' d\vec{l}' = \frac{\mu_0 I}{4\pi r^2}\oint (\hat{r}\cdot\vec{r}')d\vec{l}'$$

$$\oint (\hat{r}\cdot\vec{r}')d\vec{l}' = -\hat{r}\times\int d\vec{a}' \rightarrow \text{eq. 1.108}$$

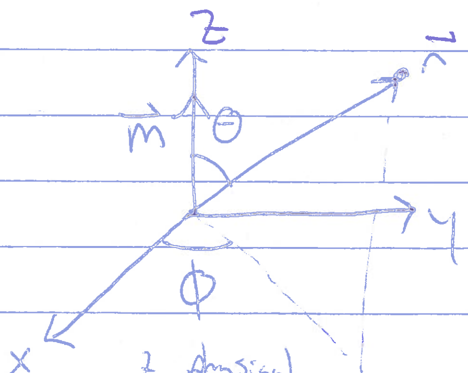
end. Wed.

minus sign flipped

→ magnetic dipole moment

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}, \quad \vec{m} \equiv I\int d\vec{a} = I\vec{a}$$

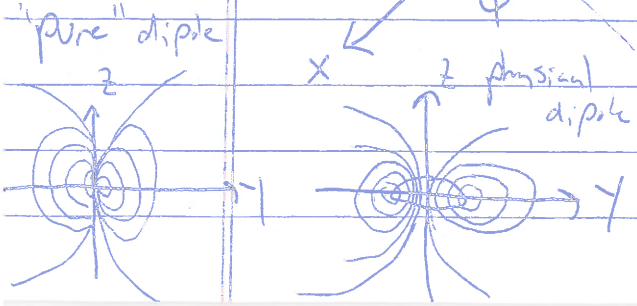
- $\vec{a} \equiv$ area enclosed by the loop (right hand rule)



$$\vec{A}_{dip} = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi} \quad \text{(same direc as current)}$$

end Fri.

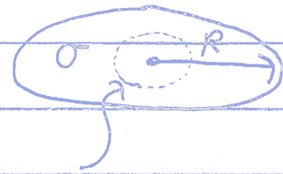
$$\vec{B}_{dip} = \nabla\times\vec{A}$$



$$= \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

→ identical to $\vec{E}_{dip}$ (form)

5.37:



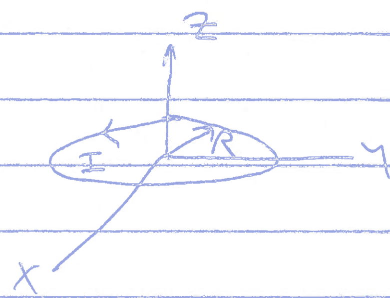
infinitesimal ring → $m = I\pi r^2$

$K = \sigma v$
$I = K \cdot dl_\perp$

$$dI = \sigma v\, dr = \sigma \omega r\, dr, \quad m = \int_0^R \sigma \omega r\, dr\, \pi r^2$$

$$= \boxed{\pi \sigma \omega R^2/4 = m}$$

5.35:



$$\vec{m} = I\vec{a} = I\pi R^2 \hat{z}$$

$$\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{I\pi R^2}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

• on z-axis $\theta = 0$, $r = z$, $\hat{r} = \hat{z}$

$$\vec{B} = \frac{\mu_0 I R^2}{2z^3}\hat{z}$$

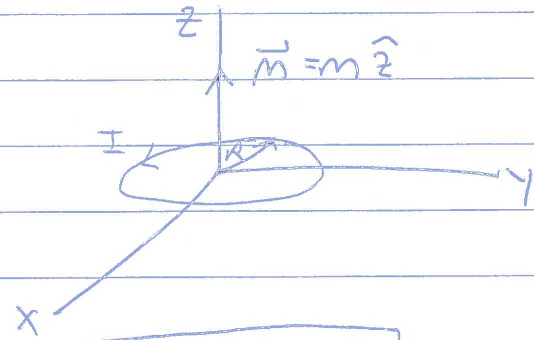
loop ←

$$\vec{B}(z) = \cancel{\frac{\mu_0 I}{4\pi}(}\ \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}, \quad z \gg R$$

$$\to \frac{\mu_0 I}{2}\frac{R^2}{z^3} \checkmark\ \hat{z}$$

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi}\frac{m\times\hat{r}}{r^2} = \frac{\mu_0 m}{4\pi}\frac{\sin\theta}{r^2}\hat{\phi}$$

(a)



- $\boxed{\vec{m} = I\cdot\pi R^2\hat{z}}$

(b) $$\vec{B}_{dip} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)$$

$$= \frac{\mu_0 \pi R^2 I}{4\pi r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)$$

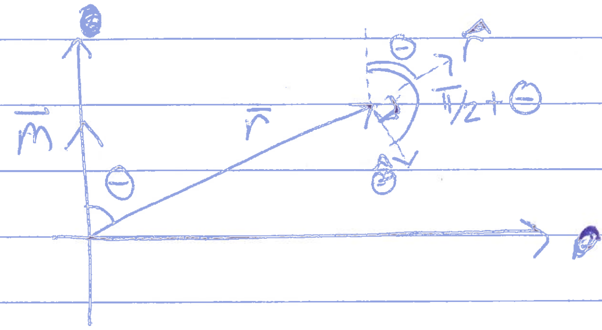
(c) z-axis $\theta = 0,\ r = z,\ \hat{r} = \hat{z}$

$$\rightarrow \vec{B}_{dip} = \frac{\mu_0 R^2 I}{4r^3}(2)\hat{r} \rightarrow \frac{\mu_0 R^2 I}{2z^3}$$ ✓

-exact solution $$\vec{B} = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}\hat{z}$$

$$z \gg R \rightarrow \frac{\mu_0 I}{2}\frac{R^2}{z^3}$$ ✓

$$B_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right]$$



$$\vec{m}\cdot\hat{r} = m\cos\theta$$
$$\vec{m}\cdot\hat{\theta} = m\cos\left(\tfrac{1}{2}\pi+\theta\right) = -m\sin\theta$$

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left\{2m\cos\theta\,\hat{r} + m\sin\theta\,\hat{\theta}\right\}$$

$$= \frac{\mu_0}{4\pi}\frac{1}{r^3}\left\{2(\vec{m}\cdot\hat{r})\hat{r} - (\vec{m}\cdot\hat{\theta})\hat{\theta}\right\}$$

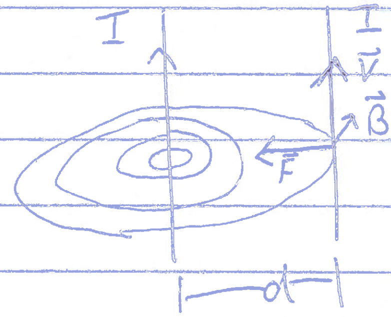
$$= \frac{\mu_0}{4\pi}\frac{1}{r^3}\left\{\underbrace{3(\vec{m}\cdot\hat{r})\hat{r} - (\vec{m}\cdot\hat{r})\hat{r}}_{2(\vec{m}\cdot\hat{r})\hat{r}}\ \underbrace{- (\vec{m}\cdot\hat{r})\hat{r} - (\vec{m}\cdot\hat{\theta})\hat{\theta}}_{-\vec{m}}\right\}$$

$$\boxed{\vec{B}_{dip} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left\{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right\}}$$

Exam I review (Mag. Statics)

Lorentz force law: $\vec{F}_{mag} = q(\vec{v} \times \vec{B})$

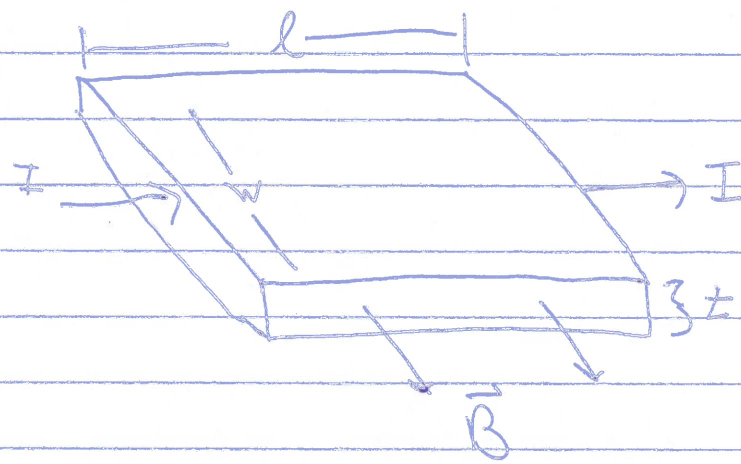
$\vec{F} = q[\vec{E} + (\vec{v} \times \vec{B})]$ $I = \lambda \vec{v}$ $\lambda = q/\ell$



$$\vec{F}_{mag} = \int I(d\vec{\ell} \times \vec{B}) = I\ell B = I\ell\left(\frac{\mu_0 I}{2\pi d}\right)$$

$$\frac{f}{\ell} = \frac{\mu_0}{2\pi}\frac{I^2}{d}$$

- Hall effect:



$\downarrow \vec{F}_{mag}$ $\uparrow \vec{F}_e$

- positive charges down, negative up.
- find the potential difference between top + bottom

$$qvB = qE \rightarrow \text{equilibrium}$$

$$\Rightarrow E = vB, \quad V = E \cdot t = \boxed{vBt = V}$$

$\rightarrow$ bottom higher pot.

- negative charges move left $\rightarrow$ deflected down
  $\rightarrow$ top plate higher potential

$$dW_{mag} = \vec{F}_{mag} \cdot d\vec{l} = q(\vec{v} \times \vec{B}) \cdot \vec{v}dt = 0$$

$$d\vec{l} = \vec{v}dt \qquad (\vec{v} \times \vec{B}) \perp \vec{v}$$

- Steady Currents: $\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \rightarrow = 0$

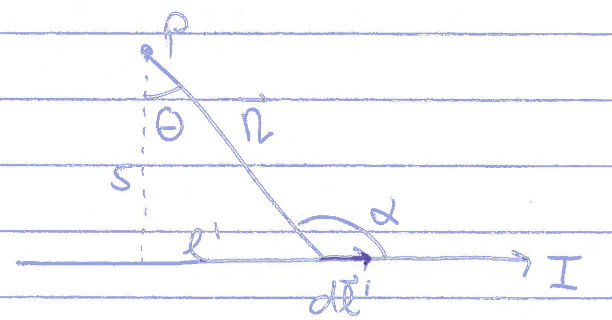
$$\nabla \cdot \vec{J} = 0, \quad \rho \rightarrow \text{constant}$$

- Biot-Savart law

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

$d\vec{l}' \times \hat{\mathscr{r}} = dl' \sin\alpha$

$\rightarrow$ out of page



$$dl' \sin\alpha = dl' \cos\theta$$

$$l' = s\tan\theta, \quad dl' = \frac{s}{\cos^2\theta} d\theta$$

$$s = \mathscr{r}\cos\theta, \quad \frac{1}{\mathscr{r}^2} = \frac{\cos^2\theta}{s^2}$$

$$B = \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \overset{1/\mathscr{r}^2}{\left(\frac{\cos^2\theta}{s^2}\right)} \overset{dl'}{\left(\frac{s}{\cos^2\theta}\right)} \overset{\cos\theta}{\cos\theta}\, d\theta$$

$$= \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \cos\theta\, d\theta = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1); \quad \theta_2 = \frac{\pi}{2}, \ \theta_1 = -\frac{\pi}{2}$$

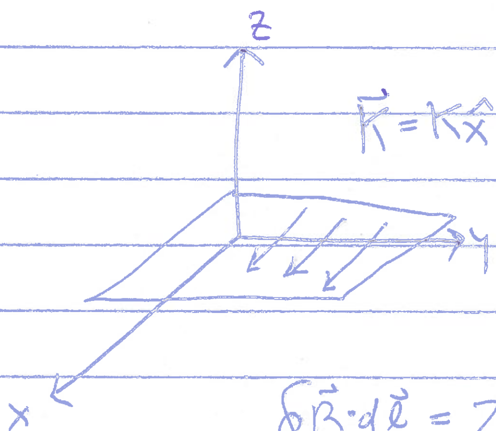
$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

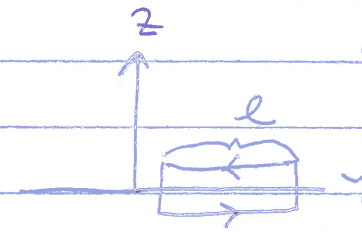
- Divergence + Curl of $\vec{B}$

$$\nabla \times \vec{B} = \mu_0 \vec{J} \Rightarrow \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$I_{enc} = \int \vec{J} \cdot d\vec{a} \; ; \; \nabla \cdot \vec{B} = 0, \text{ no Mag. monopoles}$$

$$\vec{K} \equiv \frac{d\vec{I}}{dl_\perp}$$





$$\oint \vec{B} \cdot d\vec{l} = 2Bl = \mu_0 I_{enc} = \mu_0 K l$$

$$\vec{B} = \begin{cases} (\mu_0/2) K \hat{y} \; ; \; z < 0 \\ (\mu_0/2) K \hat{y} \; ; \; z > 0 \end{cases}$$

Vector Potential: $\vec{B} = \nabla \times \vec{A}$, $\nabla \cdot \vec{A} = 0$

$$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \; ; \; \nabla^2 A = -\mu_0 \vec{J} \; , \; \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}} d\tau'$$

→ current → 0 @ ∞

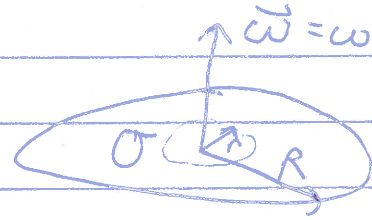
- infinite wire: $\vec{A} = A(s)\,\hat{z}$

$$\vec{B} = \nabla \times \vec{A} = -\frac{\partial A}{\partial s} \hat{\phi} = \frac{\mu_0 I}{2\pi s} \hat{\phi} = \vec{B}$$

$$\frac{\partial A}{\partial s} = -\frac{\mu_0 I}{2\pi s} \rightarrow -\frac{\mu_0 I}{2\pi} \ln(s/c)\,\hat{z}$$

$$\nabla \cdot \vec{A} = \frac{\partial A_z}{\partial z} = 0 \checkmark, \quad \nabla \times A = -\frac{\partial A_z}{\partial s}\hat{\phi} = \frac{\mu_0 I}{2\pi s}\hat{\phi} = \vec{B} \checkmark$$

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m} \times \hat{r}}{r^2}, \quad \vec{m} \equiv I \int d\vec{a}$$



$m = I\pi r^2$

$$K = \sigma v, \quad I = K l_\perp$$

$$\frac{dI}{dl_\perp} = K \Rightarrow dI = K\,dl_\perp = \sigma v\,dr$$

$v = \omega r$

$$m = \int_0^R \sigma \omega r\,dr\,\pi r^2 \Rightarrow \frac{\pi\sigma\omega R^2}{4} = m$$